

VARIETIES OF COMMUTATIVE SEMIGROUPS

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ABSTRACT. In this paper, we describe all equational theories of commutative semigroups in terms of certain well-quasi-orderings on the set of finite sequences of nonnegative integers. This description yields many old and new results on varieties of commutative semigroups. In particular, we obtain also a description of the lattice of varieties of commutative semigroups, and we give an explicit uniform solution to the word problems for free objects in all varieties of commutative semigroups.

INTRODUCTION

Varieties of semigroups have been extensively studied not only as a natural continuation of analogous investigations in the theory of groups [26], but also, more recently, because of their applications in language and automata theory (see [6, 29]).

Special attention has been given to varieties of commutative semigroups, since in analogy with groups, one could expect that these varieties should be easier to handle. Recall that a variety of commutative (abelian) groups can be described by an additional identity $x^m = 1$; and the lattice of these varieties is isomorphic to the natural numbers ordered by division (see also [13] for commutative monoids). Surprisingly, the lattice of varieties of commutative semigroups, $\mathcal{L}(\text{Com})$, turns out to be fairly complicated.

This became clear first, when Schwabauer [36] proved that, in contrast with the lattice of *all* varieties of groups, $\mathcal{L}(\text{Com})$ itself is nonmodular. Next, Burris and Nelson [4] proved that every finite partition lattice can be embedded into $\mathcal{L}(\text{Com})$, and hence it obeys no special lattice laws. In view of the later result [31], this means that $\mathcal{L}(\text{Com})$ contains *all finite lattices* as sublattices.

In a positive direction, Perkins [27] showed that every equational theory of commutative semigroups is finitely based, and hence $\mathcal{L}(\text{Com})$ is countable and has no infinite descending chains. From a general result of Evans [8, 9], it follows also that every equational theory of commutative semigroups is decidable. Yet, no explicit uniform solution to the word problems for free semigroups is known. Nelson [24] described a certain natural sublattice $\mathcal{L}_N(\text{Com})$ of $\mathcal{L}(\text{Com})$, calling it a “skeleton”, and investigated relationships between

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intervals set up by this sublattice. This extensive work was used by Almeida [2] to establish some special order properties (the so-called, better-quasi-ordering) of sublattices of $\mathcal{L}(\text{Com})$. In particular, he proved that every proper sublattice of $\mathcal{L}(\text{Com})$ consisting of subvarieties of a member of $\mathcal{L}(\text{Com})$ has no infinite antichains. He also pointed out the central role played by the sublattice $\mathcal{L}(\mathcal{N})$ of varieties of nilpotent semigroups. The structure of the latter was investigated in [18] by Korjakov. In particular, he showed that also $\mathcal{L}(\mathcal{N})$ itself contains every finite lattice as a sublattice (which, in fact, is implicit in [4]) and therefore, in his opinion, even the complete description of this special sublattice is impossible.

Another well-behaved sublattice was described by Schwabauer [37], who proved that the sublattice $\mathcal{L}_S(\text{Com})$ of varieties defined by identities of a certain special form is distributive. Nelson [24] gave another proof and showed that $\mathcal{L}_S(\text{Com})$ is in fact a maximal modular sublattice in $\mathcal{L}(\text{Com})$. Still another proof was provided in [2], but the structure of the Schwabauer sublattice remains unknown.

For other, more special and more detailed results see also [1, 5, 14, 21, 22, 23, 25, 33, 38, 40, 41, 42] and the surveys [7, 34, 35]. For descriptions of the lattices of varieties of some other comprehensive classes of semigroups see [11] and [30].

In the present paper, we describe all varieties of commutative semigroups and their lattice $\mathcal{L}(\text{Com})$ in terms of certain well-quasi-ordering on the set Γ of finite sequences of nonnegative integers. The main idea is to use known sublattices to partition $\mathcal{L}(\text{Com})$ into smaller intervals.

Section 1 is devoted to establish terminology and recall some results we use. In §2, we construct the first partition of $\mathcal{L}(\text{Com})$; in particular, we show that the Nelson sublattice $\mathcal{L}_N(\text{Com})$ determines the shape of $\mathcal{L}(\text{Com})$, since it is actually a *retract* of $\mathcal{L}(\text{Com})$. This section is closely connected with Nelson's work [24], and some lemmas in this section could be easily inferred from the results of [24]. Yet, in order to make the paper more self-contained, easier to read, and to present some possible improvements, we provide our own complete proofs.

In §3, we give the full description of the Schwabauer sublattice $\mathcal{L}_S(\text{Com})$ and the corresponding equational theories (*S*-theories). We show that there is a certain natural well-quasi-ordering \leq on the set Γ of finite sequences of nonnegative integers such that *S*-theories are in one-to-one correspondence with (and are explicitly determined by) order filters in (Γ, \leq) together with pairs of integers.

In §4, the main structure theorem is presented (Theorem 4.8). We prove that every equational theory *E* of commutative semigroups consists of the identities of some *S*-theory $E(J, m, r)$ and of a set π , called the *remainder* of *E*. The remainders are very special sets of finite character: their elements can be always described by listing a finite number of possible forms, and they are, in fact, equivalence relations whose blocks are antichains in the well-quasi-order (Γ, \leq) . It is the remainders that are responsible for that $\mathcal{L}(\text{Com})$ is not so simple and regular as it could be.

In §5, we provide an algorithm that gives explicit solutions to the word problem for free semigroups in all varieties of commutative semigroups, and which

can be even used in small hand computation. We estimate also the computational complexity of the problem by showing that, in general, to decide whether an identity e is a consequence of a finite set of commutative semigroup identities or not is *NP-hard*.

The lattice $\mathcal{L}(\text{Com})$ is described in §6. In the description, apart from usual lattice operations on integers, filters, and equivalence relations, also a certain mixed, but rather simple and natural, operation is used.

Throughout the paper, we provide many examples and applications. In the last section, we apply our results to finite semigroups. In particular, we improve Perkins' result [27] on the number of variables in an equational base of a finitely generated commutative semigroup, and characterize those varieties in $\mathcal{L}(\text{Com})$, which are generated by a finite semigroup. The latter turn out to form a sublattice of $\mathcal{L}(\text{Com})$. We show also that various finiteness properties considered for varieties are, in case of varieties of commutative semigroups, equivalent.

An integral part of the study of languages and automata is the study of pseudovarieties of semigroups (classes of *finite* semigroups closed under the formation of homomorphic images, subsemigroups, and finite direct products; see [29]). Some partial results on pseudovarieties of commutative semigroups were obtained in [2, 3]. In the subsequent article [17], we apply our results to describe all pseudovarieties of commutative semigroups and the lattice they form.

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1. PRELIMINARIES

Our approach is equational. Making use of the one-to-one correspondence between varieties and equational theories set up by the Birkhoff theorem, we study equational theories.

Assuming familiarity with basic concepts of equational logic and semigroup theory (see, e.g., [12, 39] and [6, 20, 28]) we start from the following.

1.1. We deal exclusively with equational theories of commutative semigroups; an *identity* (equation) is written in the form

$$(1) \quad x_1^{\alpha_1} \cdots x_n^{\alpha_n} = x_1^{\beta_1} \cdots x_n^{\beta_n}$$

or occasionally as a $2 \times n$ -matrix

$$(2) \quad \begin{pmatrix} \alpha_1 & \cdots & \alpha_n \\ \beta_1 & \cdots & \beta_n \end{pmatrix}$$

with $\alpha_i + \beta_i > 0$. It is *nontrivial* if $\alpha_j \neq \beta_j$ for some j . It is *regular* if $\alpha_i, \beta_i > 0$ for all i .

We axiomatize the consequence relation as follows.

Let Id be the set of matrices (2) with $\alpha_i + \beta_i > 0$. Two matrices that can be obtained from each other by permuting columns or rows are considered *equivalent*.

A set $E \subseteq Id$ of matrices (identities) is called a *closed set* (or an *equational theory of commutative semigroups*, or simply, a *theory*) if it contains all trivial matrices and the following conditions hold.

For every e of the form (2), if e is in E , then the matrices obtained from e by

- (E0) forming equivalent matrices,
 - (E1) adjoining $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ column,
 - (E2) adjoining $\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}$ column, for some i ,
 - (E3) replacing $\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}$ and $\begin{pmatrix} \alpha_j \\ \beta_j \end{pmatrix}$ columns by $\begin{pmatrix} \alpha_i + \alpha_j \\ \beta_i + \beta_j \end{pmatrix}$, for some $i \neq j$,
- are also in E . Moreover, for every e_1, e_2 , and e_3 ,

$$e_1 = \begin{pmatrix} \alpha_1 \cdots \alpha_n \\ \beta_1 \cdots \beta_n \end{pmatrix}, \quad e_2 = \begin{pmatrix} \beta_1 \cdots \beta_n \\ \gamma_1 \cdots \gamma_n \end{pmatrix}, \quad e_3 = \begin{pmatrix} \alpha_1 \cdots \alpha_n \\ \gamma_1 \cdots \gamma_n \end{pmatrix}.$$

- (E4) If e_1 and e_2 are in E , then e_3 is also in E .

Rule E1 corresponds to multiplication by a variable. E2 is substitution $x_i y_i$ for x_i . E3 is identification of variables x_i and x_j .

The least closed set containing a set of identities E is denoted by E^* . The completeness theorem in the following form holds: an identity $e \in E^*$ if and only if every commutative semigroup that is a model of E also satisfies e .

Closed sets ordered by inclusion form a complete lattice with meet $E_1 \wedge E_2 = E_1 \cap E_2$, and join $E_1 \vee E_2 = (E_1 \cup E_2)^*$. This lattice, which is dually isomorphic to the lattice of varieties of commutative semigroups, will be denoted by \mathcal{L}_{Com} .

The following well-known result is due to Perkins [27].

1.2. Proposition. *Every closed set $E \subseteq Id$ is finitely generated. Consequently, \mathcal{L}_{Com} is countable and has no infinite ascending chains.*

1.3. The following notations and conventions are used throughout the paper without further mention.

The letter e denotes always an identity of the form (1). For simplicity, we write sometimes $e = (\alpha_1 \cdots \alpha_n, \beta_1 \cdots \beta_n)$ or, if we wish to refer to the corresponding sequences, $e = (\mathbf{a}, \mathbf{b})$, where $\mathbf{a} = (\alpha_1, \dots, \alpha_n)$ and $\mathbf{b} = (\beta_1, \dots, \beta_n)$.

Identities of the form $tx = x$ and $tx = t$, where t is a term and x a variable not occurring in t , we write briefly $t = 1$ and $t = 0$, respectively.

By $E_\emptyset = \emptyset^*$ we denote the theory of all commutative semigroups, i.e., the least element in \mathcal{L}_{Com} . For maximum, minimum, the greatest common divisor, and the least common multiple of two or more integers we write \max , \min , \gcd , and lcm , respectively. The notion retract is used in an algebraic sense, i.e., a sublattice \mathcal{L}_0 of the lattice $\mathcal{L} = (L, \vee, \wedge)$ is called a *retract* if there is a (\vee, \wedge) -epimorphism ϕ of \mathcal{L} onto \mathcal{L}_0 fixing all the points of \mathcal{L}_0 .

1.4. We shall need also the following fundamentals of the theory of quasi-orderings.

A relation \leq on a set Γ is called a *quasi-order* if it is reflexive and transitive. Elements a, b satisfying $a \leq b \leq a$ are said to be equivalent. Identifying equivalent elements yields a partial order associated with \leq , which often helps

to visualize the structure of the original quasi-order. An *antichain* in a quasi-ordered set is a subset of pairwise noncomparable elements. A *chain* is a subset in which every two elements are comparable, but no two distinct elements are equivalent. An element $b \in \Gamma$ is *minimal* if $a \leq b$ implies $b \leq a$.

An *order filter*, or simply, a *filter* in a quasi-ordered set (Γ, \leq) is a set J with the property $a \in J$ and $a \leq b$ implies $b \in J$ (the name *terminal segment* is also in use). The set of all filters in the quasi-ordered set Γ forms a complete sublattice of the power set of Γ (whence, this lattice is, in particular, distributive).

The following result is well known and easy to verify.

1.5. Proposition. *For a quasi-ordered set Γ the following are equivalent.*

- (i) *Every nonempty subset of Γ has at least one and only finitely many nonequivalent minimal elements.*
- (ii) *Γ has no infinite antichains or descending chains.*
- (iii) *Every filter J in Γ is finitely generated (i.e., there is a finite set F such that $J = \{a : b \leq a, \text{ for some } b \in F\}$).*

A quasi-ordered set Γ satisfying the equivalent conditions of Proposition 1.5 is said to be *well-quasi-ordered*. The filters in a well-quasi-ordered set are especially easy to handle because of condition (iii). This condition means actually that every filter is generated and determined uniquely by its minimal elements.

It is also not difficult to observe that the property of being well-quasi-ordered is preserved under many operations, such as taking subsets, extending order, etc. In the sequel, we refer only to taking subsets and extending order, and in addition, to the following result due to Higman [15].

1.6. Proposition. *Let (Q, \leq) be a well-quasi-ordered set, and for finite sequences \mathbf{a} and \mathbf{b} of elements of Q define $\mathbf{a} \leq_m \mathbf{b}$ if some subsequence of \mathbf{b} majorizes \mathbf{a} term by term. Then, the space $Q^{<\omega}$ of all finite sequences over Q is itself well-quasi-ordered.*

For other transfer properties and more details on quasi-orderings consult [19] and the references given therein.

1.7. There are many special symbols introduced throughout the paper. For quick reference we list those used many times (and not mentioned in this section), with short explanation followed by the subsection number of the first appearance.

\equiv_N	a partition of \mathcal{L}_{Com} corresponding to the sublattice \mathcal{L}_N , 2.2.
$\mathcal{I}_N(k, m, r)$	an interval of \mathcal{L}_{Com} in the partition \equiv_N , 2.2.
N1–N4	the conditions in Lemma 2.3.
$E(k, m, r)$	the least theory in $\mathcal{I}_N(k, m, r)$, 2.7 and Lemma 2.8.
\mathcal{L}_N	the sublattice of theories $E(k, m, r)$, 2.7.
$G(k, m, r)$	the greatest theory in $\mathcal{I}_N(k, m, r)$, 2.13.
\mathcal{L}_S	the sublattice of S -theories, 3.1.
Γ, Γ_+	the set of finite sequences of nonnegative (resp. positive) integers, 3.4.

\mathcal{O}	the set of operations on the set Γ generated by the operations g_{ij} and h , 3.4.
$\mathbf{a} \leq \mathbf{b}$	the well-quasi-ordering on the set Γ defined in §3.5.
$\Gamma(e)$	the sequence associated with an S -identity e , 3.8.
$\Gamma(E)$	the set of sequences associated with S -identities in E , 3.8.
$[\mathbf{a}_1, \dots, \mathbf{a}_n]$	the filter in (Γ, \leq) generated by the sequences $\mathbf{a}_1, \dots, \mathbf{a}_n$, 3.10.
$[(m)]$	the filter generated by the one-element sequence (m) , 3.10.
$k(J), m(J)$	integers associated with a filter J in (Γ, \leq) , 3.11.
$E(J, m, r)$	a theory defined in §3.11; every S -theory is of this form, see Theorem 3.12.
$B(J, m, r)$	the normal base for $E(J, m, r)$, 3.14.
$d(J)$	an integer associated with the fundamental antichain of the filter J , 3.14.
\equiv_S	a partition of \mathcal{L}_{Com} corresponding to the sublattice \mathcal{L}_S , 4.1.
$\mathcal{I}_S(J, m, r)$	an interval of \mathcal{L}_{Com} in the partition \equiv_S , 4.1.
$\pi 0 - \pi 4$	the conditions characterizing remainders, 4.7.
$E(J, m, r, \pi)$	an arbitrary theory of \mathcal{L}_{Com} , Theorem 4.8.
$\mathcal{L}(J, m, r)$	the lattice of the remainders isomorphic to $\mathcal{I}_S(J, m, r)$, 4.10.
θ, ρ	the least and the greatest elements in $\mathcal{L}(J, m, r)$, 4.12.
$G(J, m, r)$	the greatest theory in the interval $\mathcal{I}_S(J, m, r)$, 4.12.
$E3^+, E1^\times$	additional rules of deduction, 5.1.
$C(J, \pi)$	the union of those blocks in π , which have a point in common with J , 6.1.
$E(\mathbf{S}), E(\mathcal{V})$	the equational theories of the semigroup \mathbf{S} and the variety \mathcal{V} , respectively, 7.1.
$k(\mathbf{S}), \dots, r(\mathcal{V})$	parameters given by $E(\mathbf{S})$ and $E(\mathcal{V})$, 7.1.
$Id_n(\mathbf{S})$	the identities of $E(\mathbf{S})$ in at most n variables, 7.4.

2. FIRST PARTITION

In this section, using the approach of [24], we partition the lattice \mathcal{L}_{Com} into intervals and show that this partition is actually a congruence relation on \mathcal{L}_{Com} . This, while done in a quite elementary way, gives the first insight into the structure of \mathcal{L}_{Com} , and will be used later for obtaining more advanced results.

2.1. Lemma. *If E contains some nontrivial identity e , then there exist $k, r > 0$ such that*

$$(3) \quad x^{k+r} = x^k$$

is a consequence of E , and if $x^{k_1+r_1} = x^{k_1}$ (with $k_1, r_1 > 0$) is another consequence of E , then $k \leq k_1$ and r divides r_1 .

Proof. First note that some identity of the form (3) must be in E^* . To see it, it is enough to identify all variables in e . This works, unless $\sum \alpha_i = \sum \beta_i$. In such a case, to get an inequality, we first choose j such that $\alpha_j \neq \beta_j$ and then substitute x_j^2 for x_j .

Now, for k minimal such that (3) is in E^* for some $r > 0$, choose r minimal. We show that such k and r are as claimed. Suppose, to the contrary,

that

$$(4) \quad x^{m+s} = x^m$$

is in E^* for some $s > 0$ not divisible by r . By assumption $m \geq k$. We may assume also that $0 < s < r$, because we can use (3) to reduce s modulo r . Now, from (3) we have

$$(5) \quad x^{k+\alpha r} = x^k$$

for any $\alpha \geq 0$. Replacing here α by $\alpha - 1$, and multiplying it by x^{r-s} we obtain

$$(6) \quad x^{k+\alpha r-s} = x^{k+(r-s)}$$

for any $\alpha \geq 1$. In turn, multiplying (4) by $x^{k+\alpha r-m-s}$ we get

$$(7) \quad x^{k+\alpha r} = x^{k+\alpha r-s}$$

$(k + \alpha r \geq m + s)$. Comparing (5), (6), and (7), yields

$$x^{k+(r-s)} = x^k$$

which contradicts the minimality of r .

2.2. Using Lemma 2.1, we can partition \mathcal{L}_{Com} into equivalence classes. Following Nelson [24], we add however one more parameter.

Given $k, r > 0$ and m , with $k \geq m \geq 0$, by $\mathcal{J}_N(k, m, r)$ we denote the class of those theories E in \mathcal{L}_{Com} for which k and r are minimal positive integers given by Lemma 2.1, and m is the least nonnegative integer such that

$$(8) \quad x^{m+r}y^k = x^m y^k$$

is in E . Clearly, for $m \geq k$, (8) is a consequence of (3), and that is why $m \leq k$.

The equivalence relation on \mathcal{L}_{Com} given by the definition above will be denoted by \equiv_N : two theories are N -equivalent if they belong to the same class $\mathcal{J}_N(k, m, r)$ (note, that under this definition an additional N -equivalence class is $\{E_\emptyset\}$ consisting of the theory E_\emptyset alone).

2.3. **Lemma.** *Let $E \in \mathcal{J}_N(k, m, r)$. If $e \in E$, then the following conditions hold.*

- (N1) *If $\sum \alpha_i \neq \sum \beta_i$, then both $\sum \alpha_i, \sum \beta_i \geq k$.*
- (N2) *If $\sum \alpha_i = \sum \beta_i$, then for every j such that $\alpha_j \neq \beta_j$ both $(\alpha_j + \sum \alpha_i), (\beta_j + \sum \beta_i) \geq k$.*
- (N3) *For every i , $\alpha_i \equiv \beta_i \pmod{r}$.*
- (N4) *For every i , if $\alpha_i \neq \beta_i$, then both $\alpha_i, \beta_i \geq m$.*

Proof. In view of Lemma 2.1 and the argument used in the first part of the proof of that lemma, N1 and N2 are obvious.

For N3, note that in view of Lemma 2.1, r divides $(\sum \alpha_i - \sum \beta_i)$. In turn, if we substitute x_i^2 for x_i , we get that r divides also $((\sum \alpha_i - \sum \beta_i) + (\alpha_i - \beta_i))$. Hence, r divides $(\alpha_i - \beta_i)$, as required.

To prove N4, suppose that $\alpha_j < \beta_j$ and $\alpha_j < m$. Then, identifying all variables in e except x_j , we get

$$x^{\alpha_j} y^\alpha = x^{\beta_j} y^\beta$$

where, by N3, $\alpha \equiv \beta \pmod{r}$. Multiplying this by y^s , where s is chosen so that $\alpha + s$, $\beta + s \geq k$, and both are equal to k modulo r , and then using (3), we obtain

$$x^{\alpha_j} y^k = x^{\beta_j} y^k.$$

We may assume that $\alpha_j = m - 1$ (multiplying the identity by a suitable factor, if necessary). Then, using (8), β_j (which is equal to α_j modulo r) may be reduced to $m + r - 1$, thus contradicting the minimality of m .

The conditions in Lemma 2.3 play a crucial role in this paper and in the sequel will be referred to as conditions N1–N4.

2.4. Lemma. *Let E be a nonempty set of nontrivial identities. Define*

$k_0 =$ the greatest k such that N1 and N2 hold for every e in E .

$m_0 =$ the greatest m such that N4 holds for every e in E .

$r_0 =$ the greatest r such that N3 holds for every e in E (= the greatest common divisor of all $\alpha_i - \beta_i$).

Then, $E^ \in \mathcal{J}_N(k_0, m_0, r_0)$.*

Proof. Choose k, m, r so that $E^* \in \mathcal{J}_N(k, m, r)$.

By Lemma 2.3, $k \leq k_0$, $m \leq m_0$, and r divides r_0 .

On the other hand, k_0, m_0 , and r_0 as operators on sets of identities are invariant under adjoining consequences to the set (routine checking the rules E0–E4 is left to the reader). In particular, since (3) and (8) are in E^* , $k \geq k_0$, $m \geq m_0$, and r_0 divides r .

Remark. The numbers k_0, m_0 , and r_0 in the lemma above are denoted in [24] by $U(E)$, $V(E)$, and $D(E)$, respectively.

Note also that Lemma 2.4 shows, in particular, that all the classes $\mathcal{J}_N(k, m, r)$ are nonempty.

2.5. Lemma. *If $E_1 \in \mathcal{J}_N(k_1, m_1, r_1)$ and $E_2 \in \mathcal{J}_N(k_2, m_2, r_2)$, then*

$$E_1 \cap E_2 \in \mathcal{J}_N(k_3, m_3, r_3),$$

where $k_3 = \max(k_1, k_2)$, $m_3 = \max(m_1, m_2)$, and $r_3 = \text{lcm}(r_1, r_2)$, and

$$E_1 \vee E_2 \in \mathcal{J}_N(k_4, m_4, r_4),$$

where $k_4 = \min(k_1, k_2)$, $m_4 = \min(m_1, m_2)$, and $r_4 = \text{gcd}(r_1, r_2)$.

The above formulas remain valid for infinite joins, and for infinite meets, as well, provided both corresponding maxima and the least common multiple exist; if not, then the meet is equal to E_\emptyset .

Proof. Note that as a consequence of (3) and (8) for $k = k_1$, $m = m_1$, and $r = r_1$ (or, respectively, k_2, m_2 , and r_2) we can get (3) and (8) for $k = k_3$, $m = m_3$, and $r = r_3$. Hence, these are in $E_1 \cap E_2$. Combining this with Lemma 2.4 yields the first statement. The statement for $E_1 \vee E_2$ is a direct consequence of Lemma 2.4. The arguments hold in the infinite case as well.

As the main corollary of Lemma 2.5 we have the following.

2.6. Proposition. *The relation \equiv_N is a congruence relation on \mathcal{L}_{Com} , and every equivalence class $\mathcal{J}_N(k, m, r)$ is an interval of \mathcal{L}_{Com} .*

Proof. The first statement is immediate from Lemma 2.5. For the second it is enough to observe that if $E_1 \subseteq E \subseteq E_2$ and $E_1, E_2 \in \mathcal{J}_N(k, m, r)$ then $E_1 \cap E, E \vee E_2 \in \mathcal{J}_N(k, m, r)$, and use Lemma 2.5.

2.7. Following Nelson [24], for $k \geq m$, let $E(k, m, r)$ be the theory generated by (3) and (8) (in [24] it is denoted by $\Omega_{m,k,r}^*$). By Lemma 2.4, $E(k, m, r) \in \mathcal{J}_N(k, m, r)$, and it is actually the least element in the interval $\mathcal{J}_N(k, m, r)$. Thus, the set \mathcal{L}_N consisting of all the theories $E(k, m, r)$ with E_\emptyset adjoined is a set of representatives for \equiv_N . In order to see that it is also a sublattice of \mathcal{L}_{Com} we need the following explicit solution for word problems for the corresponding free semigroups.

2.8. **Lemma.** *A nontrivial e belongs to $E(k, m, r)$ if and only if there are i and j (not necessarily distinct) such that $\alpha_i, \beta_j \geq k$ and conditions N3 and N4 are satisfied.*

Proof. First, observe that the existence of j with $\alpha_j \geq k$ is invariant under the rules E0–E4. This and Lemma 2.3(N3–N4) yield the “only if” part.

For the “if” part, let e satisfy the conditions given in the lemma. Then, using (8) we get

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} = x_1^{\gamma_1} \cdots x_n^{\gamma_n},$$

where $\gamma_i \geq k$, whenever $\alpha_i \geq m$, and $\gamma_i = \alpha_i$, otherwise. Moreover, $\gamma_i \equiv \alpha_i \pmod{r}$. The same argument holds for $x_1^{\beta_1} \cdots x_n^{\beta_n}$. Hence, using (3) completes the proof.

The lemma above appears in [24] as Theorem 2.1. The next lemma together with the fact that $E(k, m, r) \in \mathcal{J}_N(k, m, r)$ covers Theorems 1.1, 2.2, 2.3, and remarks in §2.1 of [24]. We state it as a lemma here, because we are going to obtain a much more general result (Theorem 6.2).

2.9. **Lemma.** *Let $E_1 = E(k_1, m_1, r_1)$ and $E_2 = E(k_2, m_2, r_2)$. Then the following hold.*

- (i) $E_1 \supseteq E_2$ if and only if $k_1 \leq k_2$, $m_1 \leq m_2$, and r_1 divides r_2 .
- (ii) $E_1 \cap E_2 = E(k_3, m_3, r_3)$, where $k_3 = \max(k_1, k_2)$, $m_3 = \max(m_1, m_2)$, and $r_3 = \text{lcm}(r_1, r_2)$.
- (iii) $E_1 \vee E_2 = E(k_4, m_4, r_4)$, where $k_4 = \min(k_1, k_2)$, $m_4 = \min(m_1, m_2)$, and $r_4 = \text{gcd}(r_1, r_2)$.

As in Lemma 2.5, the above formulas remain valid for infinite joins and meets. Thus \mathcal{L}_N is a complete sublattice of \mathcal{L}_{Com} .

Proof. (i) and (ii) are easy consequences of Lemma 2.8. For (iii), we infer: by Lemma 2.5, $E_1 \vee E_2 \in \mathcal{J}_N(k_4, m_4, r_4)$, by (i), $E_1, E_2 \subseteq E(k_4, m_4, r_4)$, and in consequence, $E_1 \vee E_2 \subseteq E(k_4, m_4, r_4)$. Since $E(k_4, m_4, r_4)$ is the least element in $\mathcal{J}_N(k_4, m_4, r_4)$, the result follows.

Now, combining Lemmas 2.9 and 2.5, we get a result showing that \mathcal{L}_N , in fact, determines to some extent “the shape” of \mathcal{L}_{Com} .

2.10. **Proposition.** *For every $E \in \mathcal{L}_{\text{Com}}$ define $\varphi(E)$ as the greatest element in \mathcal{L}_N contained in E . Then φ is a homomorphism of \mathcal{L}_{Com} onto \mathcal{L}_N . In particular, \mathcal{L}_N is a retract of \mathcal{L}_{Com} , and $\mathcal{L}_N \cong \mathcal{L}_{\text{Com}} / \equiv_N$.*

2.11. Lemma 2.8 can be used also to determine the cardinality of the n -generated free semigroups $F_n(E(k, m, r))$ in the varieties corresponding to theories $E(k, m, r)$. We can distinguish two types of words $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ in such semigroups: those with all $\alpha_i < k$ (there are $k^n - 1$ of them), and those

with some $\alpha_i \geq k$. In the latter case all exponents except one can be reduced modulo r to be less than $m+r$, and it is not hard to see that there are precisely $(m+r)^n - m^n$ distinct words. Hence we have

2.12. Corollary. *The cardinality $|\mathbb{F}_n(E(k, m, r))| = k^n + (m+r)^n - m^n - 1$ for all $n \geq 0$.*

It is also not difficult to describe the greatest elements in each $\mathcal{J}_N(k, m, r)$. Let us denote it by $G(k, m, r)$. Then, by Lemmas 2.3 and 2.4, we have

2.13. Corollary. *For a nontrivial e , $e \in G(k, m, r)$, if and only if conditions N1–N4 are satisfied.*

Using this we prove

2.14. Proposition. *Given $k, r > 0$ and $k \geq m \geq 0$ let E consist of the following identities*

$$(9) \quad y^m x_1 \cdots x_{k-m} = y^{m+r} x_1 \cdots x_{k-m}$$

and, in addition, for every $u \geq m$ and such that $2u + r < k$

$$(10) \quad z^u y^{u+r} x_1 \cdots x_q = z^{u+r} y^u x_1 \cdots x_q,$$

where $q = \max(0, k - 3u - r)$.

Then, $E^ = G(k, m, r)$.*

Proof. $E^* \in \mathcal{J}_N(k, m, r)$ by Lemma 2.4. In view of Corollary 2.13 we need only show that every nontrivial e satisfying N1–N4 belongs to E^* . According to N1–N2 there are two possibilities for e .

1. If $\sum \alpha_i, \sum \beta_i \geq k$, then (9) applies and the claim is obvious.

2. If $\sum \alpha_i = \sum \beta_i < k$, then choose i such that $\alpha_i \neq \beta_i$. If, for instance, $\alpha_i < \beta_i$, then in this case, there is also j such that $\alpha_j > \beta_j$. Let $u = \min(\alpha_i, \beta_j)$. We wish to show first that now (10) can be applied.

Assume, for instance, that $u = \alpha_i$. Then, since $\alpha_j \equiv \beta_j \pmod{r}$ and $\alpha_j > \beta_j$, $\alpha_j \geq u + r$. It follows that $2u + r \leq \sum \alpha_i < k$. Also, by virtue of N4, $u \geq m$. It remains to show that $\sum \alpha_i \geq 2u + r + q$. To this end it is enough to observe that the latter is equivalent to $\sum \alpha_i \geq k - u$, which is an immediate consequence of N2.

Repeated application of (10) proves that also in case 2, $e \in E^*$.

2.15. By Corollary 2.13, $G(k_1, m_1, r_1) \cap G(k_2, m_2, r_2) = G(k_3, m_3, r_3)$, where k_3, m_3 , and r_3 are as in Lemma 2.5. But $G(k_1, m_1, r_1) \vee G(k_2, m_2, r_2)$ may fail to equal $G(k_4, m_4, r_4)$. For example, using Proposition 2.14, it is easy to find that $G(2, 1, 1) = \{x^2 = x^3\}^*$, $G(3, 0, 1) = \{xyz = 0\}^*$, while $G(2, 0, 1) = \{xy = 0\}^* \neq \{x^2 = x^3, xyz = 0\}^*$ (the latter inequality one may easily prove directly or check using Theorem 3.12; see §3.17).

2.16. To close the section, let us have a look at the content of some intervals $\mathcal{J}_N(k, m, r)$. Obviously, $E(1, 0, 1) = \{x = y\}^*$ is the greatest element of \mathcal{L}_{Com} , and therefore $\mathcal{J}_N(1, 0, 1)$ is a class consisting of only $\{x = y\}^*$. Further, we have $E(1, 0, r) = G(1, 0, r) = \{x^r = 1\}^*$. Hence, for every $r \geq 1$, $\mathcal{J}_N(1, 0, r)$ consists only of one element: the equational theory of *abelian groups* of exponent r . Those with r prime form an infinite set of coatoms in \mathcal{L}_{Com} (the remaining two are $E(1, 1, 1)$ and $G(2, 0, 1)$ below).

Also, $E(k, k, r) = G(k, k, r) = \{x^{k+r} = x^k\}^*$, and the classes $\mathcal{F}_N(k, k, r)$ are singletons. In particular, $E(k, k, 1)$ are theories of *aperiodic semigroups*, and $E(1, 1, 1)$ is the theory of *semilattices*.

$E(k, 0, 1) = \{x^k = 0\}^*$ are theories of *nil-semigroups*. By Proposition 2.14, $G(k, 0, 1) = \{x_1 \cdots x_k = 0, \dots\}^*$, where dots mean the possibility of some further identities in accordance with the proposition. (In particular, $G(2, 0, 1) = \{xy = 0\}^*$, $G(3, 0, 1) = \{xyz = 0\}^*$, but $G(4, 0, 1) = \{xyzzt = 0, xy^2 = x^2y\}^*$). Hence, $\mathcal{F}_N(k, 0, 1)$ consists of more than one element, and as a matter of fact, every such interval is infinite. For example, $\mathcal{F}_N(2, 0, 1)$ consists of the chain of the theories $\{x^2 = 0, x_1 \cdots x_k = 0\}^*$.

Finally, $E(2, 0, r) = \{x^r y^2 = y^2\}^*$, $G(2, 0, r) = \{x^r yz = yz\}^*$, and $\mathcal{F}_N(2, 0, r)$ are again infinite. We shall see later (§3.17) that $\mathcal{F}_N(k, m, r)$ always either consists of one element, whenever $k = m$ or $k = 1$, or is infinite, otherwise.

Note also that, in view of Lemma 2.5, \mathcal{L}_{Com} has no atoms, and the least element E_\emptyset is meet irreducible.

3. SCHWABAUER SUBLATTICE

In order to obtain a finer partition, we would like to apply the successful approach of §2 to the sublattice \mathcal{L}_S of the Schwabauer theories [36]. The problem is however that no description of \mathcal{L}_S is known. The aim of this section is to provide such a one.

3.1. Recall that a nontrivial identity e of the form (1) is called an *S-identity* if $\alpha_i \leq \beta_i$ for all i . A theory generated by one or more *S-identities* is called an *S-theory*. The subset \mathcal{L}_S of \mathcal{L}_{Com} consisting of all *S-theories* and the theory E_\emptyset is a sublattice of \mathcal{L}_{Com} [36]. Since every $E(k, m, r)$ is defined by *S-identities*, we have $\mathcal{L}_N \subseteq \mathcal{L}_S$.

Our description is based on the following.

3.2. **Lemma.** *Let $E \in \mathcal{F}_N(k, m, r)$, and let*

$$e = \left(\begin{matrix} \alpha_1 \cdots \alpha_n \\ \beta_1 \cdots \beta_n \end{matrix} \right), \quad f = \left(\begin{matrix} \alpha_1 \cdots \alpha_n \\ \gamma_1 \cdots \gamma_n \end{matrix} \right).$$

If e is an S-identity with $\alpha_i \leq \beta_i$, then the following are equivalent.

- (i) $e \in E$.
- (ii) *Some f with $\gamma_i = \alpha_i$ whenever $\alpha_i < m$, and $\gamma_i \geq k$, $\gamma_i \equiv \alpha_i \pmod{r}$, otherwise, is in E .*
- (iii) *All f with $\gamma_i = \alpha_i + k_i r$, for all $k_1, \dots, k_n \geq 0$, such that $k_i = 0$ whenever $\alpha_i < m$, are in E .*
- (iv) *For some j with $\alpha_j \geq m$, f with $\gamma_j = \alpha_j + r$, and $\gamma_i = \alpha_i$, otherwise, is in E .*

Proof. First we show that (i)–(iii) are equivalent.

(i) \Rightarrow (ii). Suppose that $\beta_i = \alpha_i + k_i r$. Then, multiplying e by $x_i^{k_i r}$ for all i , and comparing the resulting identity with e itself, we deduce that $f \in E$ with $\gamma_i = \alpha_i + 2k_i r$. Repeating this several times and using (3), we obtain (ii).

(ii) \Rightarrow (iii). It is enough to multiply f in (ii) by $x_i^{k_i r}$ for all i , and again compare the resulting identity with f itself, applying (3).

(iii) \Rightarrow (i). Trivial.

Since also (iii) \Rightarrow (iv) trivially holds, to complete the proof, it is enough to show, e.g., (iv) \Rightarrow (ii).

(iv) \Rightarrow (ii). Let j be fixed. Multiplying f in (iv) by x_j^r , and using an argument analogous to that in (i) \Rightarrow (ii), we obtain that f with $\gamma_j \geq k$, $\gamma_j \equiv \alpha_j \pmod{r}$, and $\gamma_i = \alpha_i$, otherwise, is in E . Now, (ii) follows easily from (8).

3.3. The lemma above shows that every sequence $(\alpha_1, \dots, \alpha_n)$ gives rise to a class of S -identities that are all equivalent in $\mathcal{S}_N(k, m, r)$. Moreover, in view of (iv), if $m = 0$, then $\alpha_i = 0$ play no role and can be omitted. Also the arrangement of elements plays no role. This suggests a possibility of expressing the consequence relation in terms of a relation on finite sequences (rather than matrices), and we proceed to describe a relevant order.

3.4. Let Γ be the set of finite sequences $(\alpha_1, \dots, \alpha_n)$ of nonnegative integers such that at least one $\alpha_i \neq 0$. The following set \mathcal{O} of operations on the set Γ plays a crucial role in the sequel. Given a sequence $(\alpha_1, \dots, \alpha_n) \in \Gamma$, we define unary operations g_{ij} and h on Γ by

$$g_{ij}((\alpha_1, \dots, \alpha_n)) = (\alpha_1, \dots, \alpha_i + \alpha_j, \dots, \alpha_n)$$

(where $(\alpha_1, \dots, \alpha_i + \alpha_j, \dots, \alpha_n)$ is a sequence obtained from $(\alpha_1, \dots, \alpha_n)$ by replacing α_i by $\alpha_i + \alpha_j$, and deleting α_j , provided $i, j \leq n$ and $i \neq j$; otherwise, g_{ij} is the identity operation on Γ), and

$$h((\alpha_1, \dots, \alpha_n)) = (\alpha_1, \dots, \alpha_n, 1).$$

By \mathcal{O} we denote the least composition closed set of operations on Γ containing all g_{ij} , and h . Note that, in particular, the identity operation g_{ii} , also called *trivial*, is in \mathcal{O} .

3.5. Now, a quasi-order relation \leq on Γ is defined as follows.

$$(\alpha_1, \dots, \alpha_n) \leq (\beta_1, \dots, \beta_m)$$

if there exists $f \in \mathcal{O}$ such that $f((\alpha_1, \dots, \alpha_n))$ and $(\beta_1, \dots, \beta_m)$ differ at most in the arrangement of elements and the number of zeros occurring in the sequence. (In other words, $(\beta_1, \dots, \beta_m)$ can be obtained from $(\alpha_1, \dots, \alpha_n)$ by applying operations g_{ij} , h , permutations, deleting, and adjoining zeros.) Reflexivity and transitivity of \leq are immediate. It is also easy to see that two sequences are *equivalent* if and only if they differ at most in the arrangement of elements and the number of zeros. The relation has the following natural combinatorial meaning.

3.6. **Lemma.** $(\alpha_1, \dots, \alpha_n) \leq (\beta_1, \dots, \beta_m)$ if and only if there is a partition π of the set $\{1, \dots, n\}$ and a one-to-one mapping ψ from the set B_π of the blocks of π into the set $\{1, \dots, m\}$ such that for every $B \in B_\pi$,

$$\sum_{i \in B} \alpha_i \leq b_{\psi(B)}.$$

Proof. Trivial.

Observing that for $\Gamma = Q^{<\omega}$ the quasi-order \leq extends the order \leq_m defined in Proposition 1.6, we have also the following.

3.7. **Lemma.** *The relation \leq is a well-quasi-ordering on Γ .*

3.8. Now, for every S -identity e with $\alpha_i \leq \beta_i$, we define $\Gamma(e) = (\alpha_1, \dots, \alpha_n) \in \Gamma$, and for a set of identities E by $\Gamma(E)$ we denote the subset of Γ consisting of all sequences $\Gamma(e)$ corresponding to S -identities e in E . Then, the connection between S -theories and filters in the quasi-order \leq on Γ is the following.

3.9. **Lemma.** *Let S be a set of S -identities, and suppose that $S^* \in \mathcal{S}_N(k, m, r)$. Then, a nontrivial $e \in S^*$ if and only if e satisfies conditions N1–N4 and both $(\alpha_1, \dots, \alpha_n)$ and $(\beta_1, \dots, \beta_n)$ belong to the filter J in (Γ, \leq) generated by $\Gamma(S)$. In particular, $\Gamma(S^*) = J$.*

Proof. First note, that if some $\mathbf{b} = (\beta_1, \dots, \beta_m)$ belongs to the filter generated by $\Gamma(S)$, then an S -identity e with $\Gamma(e) = \mathbf{b}$ is in S^* . Indeed, if $\mathbf{a} = (\alpha_1, \dots, \alpha_n) \in \Gamma(S)$ is such that $f((\alpha_1, \dots, \alpha_n))$ is equivalent to $(\beta_1, \dots, \beta_m)$ for some $f \in \mathcal{O}$ (i.e., $\mathbf{a} \leq \mathbf{b}$), and $e_1 \in S$ such that $\Gamma(e_1) = \mathbf{a}$, then a desired identity can be obtained from e_1 by applying rules E3, E1, and E0 in accordance with how f is composed from g_{ij} and h , respectively.

Whence, to prove the “if” part of the lemma it is enough to compare the S -identities corresponding to $(\alpha_1, \dots, \alpha_n)$ and $(\beta_1, \dots, \beta_n)$ given in the form of Lemma 3.2(ii), applying Lemma 2.3 and E4.

For the “only if” part we induct on the length of the proof of e from $\Gamma(S)$. If $e \in \Gamma(S)$, then the statement is obviously true. It remains to check that if e (and f) has (have) the property in question, then so does every identity obtained by applying the rules E0–E4. This is routine and is left to the reader.

$\Gamma(S^*) = J$ by virtue of the first statement of the proof.

3.10. It follows from Lemma 3.9 that every S -theory E can be generated by a set of S -identities corresponding to a filter J ; namely, $J = \Gamma(E)$. In order to establish a one-to-one correspondence between S -theories and filters, we first make some observations. In the sequel, by $[\mathbf{a}_1, \dots, \mathbf{a}_n]$ we denote the filter generated by the sequences $\mathbf{a}_1, \dots, \mathbf{a}_n$. In particular, $[(m)]$ denotes the (principal) filter generated by the one-element sequence (m) . For $m = 0$, we define $[(0)] = [(1)] = \Gamma$.

3.11. If $E \in \mathcal{S}_N(k, m, r)$, then the one-element sequence (k) belongs to J , and k is the least with this property. Given J let us denote the least such integer by $k(J)$. Moreover, if $(\alpha_1, \dots, \alpha_n) \in J$, then in view of Lemma 2.3, and since S -identities are assumed to be nontrivial, the inequality $\max(\alpha_1, \dots, \alpha_n) \geq m$ is satisfied. The least integer among $\max(\alpha_1, \dots, \alpha_n)$, for all $(\alpha_1, \dots, \alpha_n) \in J$, we denote by $m(J)$.

Now, given m, r , and a filter J such that $m(J) \geq m$, let S be a set of S -identities corresponding to J and satisfying conditions N1–N4, with $k = k(J)$, and let us denote S^* by $E(J, m, r)$. Then, since $(k) \leq (k, m) \in J$, we have $S^* = E(J, m, r) \in \mathcal{S}_N(k, m, r)$, and in consequence the hypothesis of Lemma 3.9 holds.

Note, that for $(\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n) \in J$ conditions N1–N2 for $k = k(J)$ are always satisfied, whence they can be omitted. On the other hand, $m(J) \geq m$ means that $J \subseteq [(m)]$.

Note also, that elements of J represent distinct S -identities, and it follows, by Lemma 3.9, that $E(J_1, m_1, r_1) \supseteq E(J_2, m_2, r_2)$ implies, in particular, $J_1 \supseteq J_2$.

Combining all these remarks with Lemma 2.5, we obtain the following description of the lattice \mathcal{L}_S .

3.12. Theorem. *A theory $E \in \mathcal{L}_{\text{Com}}$ is an S -theory iff $E = E(J, m, r)$, for some $m \geq 0$, $r > 0$, and a nonempty order filter $J \subseteq [(m)]$. Then, a nontrivial $e \in E$ if and only if e satisfies conditions N3–N4 and both $(\alpha_1, \dots, \alpha_n)$, $(\beta_1, \dots, \beta_n) \in J$. Moreover, if $E_1 = E(J_1, m_1, r_1)$ and $E_2 = E(J_2, m_2, r_2)$, then the following hold.*

- (i) $E_1 \supseteq E_2$ if and only if $J_1 \supseteq J_2$, $m_1 \leq m_2$, and r_1 divides r_2 .
- (ii) $E_1 \cap E_2 = E(J_1 \cap J_2, \max(m_1, m_2), \text{lcm}(r_1, r_2))$.
- (iii) $E_1 \vee E_2 = E(J_1 \cup J_2, \min(m_1, m_2), \text{gcd}(r_1, r_2))$.

These formulas remain valid for infinite joins and meets except that if the intersection of the filters is empty or if the maximum or the common multiple in question does not exist, then the meet equals E_\emptyset .

Note, that the intersection of two filters $J_1 \cap J_2$ (of a finite number of filters) is never empty, since for $k = \max(k(J_1), k(J_2))$, we have $(k) \in J_1 \cap J_2$. Moreover, $E(k, m, r) = E([(k)], m, r)$.

3.13. Corollary. \mathcal{L}_S is a distributive sublattice of \mathcal{L}_{Com} closed under infinite joins and meets, and containing \mathcal{L}_N .

3.14. We present a finite equational base for $E(J, m, r)$ consisting only of S -identities and involving the least possible number of variables. It determines in a straightforward way the filter J and the theory $E(J, m, r)$; therefore, it is called *normal*.

A maximal antichain of minimal elements in J is called *fundamental* if it consists only of increasing sequences with no zeros. Clearly, such an antichain exists and is unique and, by remarks in §1.4, it generates the filter J . By $d(J)$ we denote the maximal length of the sequence in the fundamental antichain of J , which equals, obviously, the least number such that J can be generated by sequences of the length not exceeding n . (We refer to this number in further sections.)

Now, by $B(J, m, r)$ we denote the set of identities

$$(11) \quad x_1^{\alpha_1+r} x_2^{\alpha_2} \dots x_n^{\alpha_n} = x_1^{\alpha_1} \dots x_n^{\alpha_n},$$

for every $(\alpha_1, \dots, \alpha_n)$ in the fundamental antichain of J , and the identity (8), in addition, whenever m is not already determined by the identities (11) themselves (i.e., whenever $\alpha_1 > m$ for all sequences $(\alpha_1, \dots, \alpha_n)$ in the fundamental antichain of J).

Then, combining Theorem 3.12 with Lemma 3.7 and Proposition 1.5, we have the following.

3.15. Corollary. *Every S -theory $E(J, m, r)$ is generated by a finite set of S -identities; in particular, $B(J, m, r)$ is an equational base for $E(J, m, r)$.*

3.16. Theorem 3.12 may be applied, in particular, to theories generated by irregular identities, that is, those which have a variable occurring only on one side. It is not difficult to see that every such identity is equivalent to a pair of S -identities. For example, the identity

$$(12) \quad x^2 y^7 = x^3 y^4 z^2$$

is equivalent to

$$x^2y^7 = x^5y^8z^2, \quad x^3y^4z^2 = x^5y^8z^2$$

(just substitute xy^2z for z in (12)). In particular, the theory generated by (12) is $E([(2, 3, 4)], 0, 1)$ (note, that $(2, 7) \in [(2, 3, 4)]$). Consequently, the identity $x^3y^6 = x^4z^5$ is a consequence of (12), while, for example, $x^3y^3z^3 = x^2y^7z^3$ is not.

3.17. We check the inequality in §2.15.

$$G(2, 0, 1) = \{xy = 0\}^* = E([(1, 1)], 0, 1),$$

while

$$\{x^2 = x^3, xyz = 0\} = E([(2), (1, 1, 1)], 0, 1).$$

Also the statement in §2.16 concerning the cardinality of $\mathcal{J}_N(k, m, r)$ can be easily verified. Suppose that $k > m$ and $k > 1$. Then, for every filter $J_s^k = [(k), (k-1, \dots, k-1)]$, where $k-1$ occurs $s \geq 2$ times, we have $k(J) = k$, and the theories $E(J_s^k, m, r)$ are pairwise distinct. All of them lie in the interval $\mathcal{J}_N(k, m, r)$; whence, it is infinite.

4. GENERAL STRUCTURE THEOREM

In this section, we use the sublattice \mathcal{L}_S to partition \mathcal{L}_{Com} into smaller intervals. We follow our approach in §2. Although the corresponding partition is no longer a congruence relation, the intervals are already small enough to yield a general structure theorem for theories $E \in \mathcal{L}_{\text{Com}}$.

4.1. Given $m \geq 0$, $r > 0$, and $\emptyset \neq J \subseteq [(m)]$, let $\mathcal{J}_S(J, m, r)$ be the class of all theories $E \in \mathcal{L}_{\text{Com}}$, such that the join of all S -theories contained in E is equal to $E(J, m, r)$ (in other words, such that $E(J, m, r)$ is the greatest S -theory contained in E). The corresponding equivalence relation having as blocks the classes $\mathcal{J}_S(J, m, r)$ and $\{E_\emptyset\}$ we denote by \equiv_S .

Since (3) and (8) are S -identities, $\mathcal{J}_S(J, m, r) \subseteq \mathcal{J}_N(k(J), m, r)$. Hence, $\equiv_S \subseteq \equiv_N$. The least element in $\mathcal{J}_S(J, m, r)$ is obviously $E(J, m, r)$. Furthermore, if $E_1, E_2 \in \mathcal{J}_S(J, m, r)$ and $E_1 \subseteq E \subseteq E_2$, then all E , $E_1 \cap E_2$, $E_1 \vee E_2 \in \mathcal{J}_S(J, m, r)$ (cf. Theorem 3.12). This is also true for infinite meets and joins. Thus, we have the following.

4.2. **Proposition.** *Each equivalence class $\mathcal{J}_S(J, m, r)$ is an interval of \mathcal{L}_{Com} . Moreover, $\mathcal{J}_S(J, m, r) \subseteq \mathcal{J}_N(k(J), m, r)$.*

In view of Lemma 2.5, we have also

4.3. **Lemma.** *If $E_1 \in \mathcal{J}_S(J_1, m_1, r_1)$ and $E_2 \in \mathcal{J}_S(J_2, m_2, r_2)$, then the following hold.*

- (i) $E_1 \cap E_2 \in \mathcal{J}_S(J_1 \cap J_2, \max(m_1, m_2), \text{lcm}(r_1, r_2))$,
- (ii) $E_1 \vee E_2 \in \mathcal{J}_S(J, \min(m_1, m_2), \text{gcd}(r_1, r_2))$, for some $J \supseteq J_1 \cup J_2$.

As in Theorem 3.12, these formulas remain valid for infinite joins and meets.

In §6.1, we shall see that J in (ii) may be different from $J_1 \cup J_2$, and therefore \equiv_S is not a congruence relation.

4.4. Let $E \in \mathcal{J}_S(J, m, r)$ and e be an identity. Our aim is now to find necessary conditions for e to belong to E , analogous to those in Lemma 2.3, and such that a counterpart of Corollary 2.13 holds.

Obviously, at first, conditions N1–N4 must be satisfied, since $\mathcal{J}_S(J, m, r) \subseteq \mathcal{J}_N(k, m, r)$. It is also not difficult to observe the following.

4.5. **Lemma.** *Let $E \in \mathcal{J}_S(J, m, r)$ and let e of the form (1) be in E . Then*

- (i) $(\alpha_1, \dots, \alpha_n) \in J$ if and only if $(\beta_1, \dots, \beta_n) \in J$.
- (ii) For every i such that $\alpha_i \neq \beta_i$, both $(\alpha_1, \dots, \alpha_n, \alpha_i)$ and $(\alpha_1, \dots, \alpha_n, \beta_i) \in J$.

Proof. For (i), suppose that $(\alpha_1, \dots, \alpha_n) \in J$. Then, by Lemma 3.2(iii) an S -identity

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$$

with $\gamma_i \geq \beta_i$ is in E . By E4, the α 's can be replaced by β 's. Hence, $(\beta_1, \dots, \beta_n) \in J$, too.

For (ii), note that, by E1, $(\alpha_1 \cdots \alpha_n \beta_i, \beta_1 \cdots \beta_n \beta_i) \in E$, and by E2, $(\alpha_1 \cdots \alpha_n \alpha_i, \beta_1 \cdots \beta_n \beta_i) \in E$. Hence, by E4, $(\alpha_1 \cdots \alpha_n \alpha_i, \alpha_1 \cdots \alpha_n \beta_i) \in E$. The latter is an S -identity, and the result follows by the definition of $\mathcal{J}_S(J, m, r)$.

We note also the following.

4.6. **Lemma.** *If $e = (\mathbf{a}, \mathbf{b})$ is an S -identity with $\mathbf{a}, \mathbf{b} \notin J$, then e has a consequence $e_1 = (\mathbf{a}_1, \mathbf{b}_1)$ such that $\mathbf{a}_1 \in J$, while $\mathbf{b}_1 \notin J$.*

Proof. Assume that $\beta_i = \alpha_i + \delta_i$, $\delta_i \geq 0$, for all i , and $\delta_j > 0$ for some j . Multiplying e by suitable factors, we obtain the S -identity (\mathbf{b}, \mathbf{c}) , with $\mathbf{c} = (\beta_1 + \delta_1, \dots, \beta_n + \delta_n)$. If $\mathbf{c} \in J$, we are done; if not, we repeat the argument. Since every $(\gamma_1, \dots, \gamma_n)$ with $\gamma_j \geq k(J)$ for some j is in J , after a finite number of steps we get the result.

4.7. Now, suppose that $E \in \mathcal{J}_S(J, m, r)$. Then, we can write

$$E = E(J, m, r) \cup \pi,$$

where π is just $E \setminus E(J, m, r)$ with trivial identities of the form (\mathbf{a}, \mathbf{a}) , $\mathbf{a} \notin J$, adjoined. We shall call it the *remainder* of the theory E , since, as we shall see, it is a very special and relatively small part of E . We proceed to establish some properties of the remainder.

At first, if $(\mathbf{a}, \mathbf{b}) \in E$, then since $\mathcal{J}_S(J, m, r) \subseteq \mathcal{J}_N(k, m, r)$ for $k = k(J)$, it satisfies conditions N1–N4. Hence, if in addition, $\mathbf{a}, \mathbf{b} \in J$, then by Lemma 3.9, $(\mathbf{a}, \mathbf{b}) \in E(J, m, r)$. Consequently, in view of Lemma 4.5(i), if $(\mathbf{a}, \mathbf{b}) \in \pi$, then $\mathbf{a}, \mathbf{b} \notin J$. In particular, since E is closed, π itself is closed under E4, that is, π is transitive.

At second, if $e \in \pi$, $e = (\alpha_1 \cdots \alpha_n, \beta_1 \cdots \beta_n)$, then all $\alpha_i, \beta_i > 0$ (that is, e is regular). Indeed, if for example, $\alpha_i > \beta_i = 0$ for some i , then substitution $x_i^{k(J)}$ for x_i in e yields an identity contradicting Lemma 4.5(i). The set π can be viewed therefore as an equivalence relation on the set $\Gamma_+ \setminus J$ of those finite sequences of *positive* integers that are not in J .

Then, if $(\mathbf{a}, \mathbf{b}) \in \pi$, with $\mathbf{a} = (\alpha_1, \dots, \alpha_n)$, and $\mathbf{b} = (\beta_1, \dots, \beta_t)$, also the following conditions are satisfied.

$$(\pi 0) \quad n = t.$$

- (π_1) Conditions N1–N4 hold for $k = k(J)$.
- (π_2) For every permutation σ of elements of an n -element sequence, $(\sigma(\mathbf{a}), \sigma(\mathbf{b})) \in \pi$.
- (π_3) For every i with $\alpha_i \neq \beta_i$, $(\alpha_1, \dots, \alpha_n, \min(\alpha_i, \beta_i)) \in J$.
- (π_4) For every $f \in \mathcal{O}$, either $(f(\mathbf{a}), f(\mathbf{b})) \in \pi$ or both $f(\mathbf{a}), f(\mathbf{b}) \in J$.

Of these conditions π_0 is trivial, π_1 is for $E \in \mathcal{S}_S(J, m, r)$, π_2 is by E0, π_3 is by Lemma 4.5(ii), and π_4 follows from Lemma 4.5(i), since $(f(\mathbf{a}), (f(\mathbf{b})))$ is a consequence of (\mathbf{a}, \mathbf{b}) .

Our structure theorem states that these conditions are also sufficient for $E(J, m, r) \cup \pi$ to be a theory.

4.8. Theorem. *Given integers $m \geq 0$, $r > 0$, and a nonempty order filter $J \subseteq [(m)]$, let π be an equivalence relation on the set $\Gamma_+ \setminus J$ satisfying conditions π_0 – π_4 , and let $E(J, m, r, \pi) = E(J, m, r) \cup \pi$. Then, the following hold.*

- (i) $E(J, m, r, \pi)$ is a closed set.
- (ii) $E(J, m, r, \pi) \in \mathcal{S}_S(J, m, r)$, i.e., the S -identities in $E(J, m, r, \pi)$ are those belonging to $E(J, m, r)$.
- (iii) $(\mathbf{a}, \mathbf{b}) \in E(J, m, r, \pi)$ if and only if either both $\mathbf{a}, \mathbf{b} \in J$ and satisfy conditions N3 and N4 or both $\mathbf{a}, \mathbf{b} \notin J$ and $(\mathbf{a}, \mathbf{b}) \in \pi$.

Conversely, every $E \in \mathcal{L}_{\text{Com}}$ other than E_\emptyset is of this form, i.e., if $E \in \mathcal{S}_S(J, m, r)$, then there is the unique equivalence relation π on the set $\Gamma_+ \setminus J$ satisfying conditions π_0 – π_4 such that $E = E(J, m, r, \pi)$.

Proof. We prove (i). Since $E(J, m, r)$ is a closed set, and by the definition, if $(\mathbf{a}, \mathbf{b}) \in E(J, m, r)$, then either $\mathbf{a}, \mathbf{b} \in J$ or $\mathbf{a}, \mathbf{b} \notin J$, it is enough to check the rules E0–E4 with respect to the set π alone. We have to show that if $e = (\mathbf{a}, \mathbf{b})$ is obtained from identities in π by applying one of E0–E4, then either $e \in E(J, m, r)$ or else e satisfies conditions π_0 – π_4 . And since conditions π_0 – π_2 are obviously satisfied (for $E \in \mathcal{S}_S(J, m, r)$), it suffices to show that either $\mathbf{a}, \mathbf{b} \in J$ or e satisfies conditions π_3 and π_4 .

For E0 (i.e., if e is obtained by applying E0) this is trivial. For E2 it is enough to apply π_3 and use the fact that J is a filter, to see that both $\mathbf{a}, \mathbf{b} \in J$.

For E4, let $(\mathbf{a}, \mathbf{b}), (\mathbf{b}, \mathbf{c}) \in \pi$. Then by π_4 , either $f(\mathbf{a}), f(\mathbf{c}) \in J$ or $(f(\mathbf{a}), f(\mathbf{b})), (f(\mathbf{b}), f(\mathbf{c})) \in \pi$. In the latter case, $(f(\mathbf{a}), f(\mathbf{c})) \in \pi$, since π is an equivalence relation. It follows that π_4 holds for (\mathbf{a}, \mathbf{c}) . Also π_3 does. Indeed, denote $\mathbf{c} = (\gamma_1, \dots, \gamma_n)$. If $\alpha_i \leq \gamma_i$, then by applying π_3 for (\mathbf{a}, \mathbf{b}) and (\mathbf{b}, \mathbf{c}) , either $(\alpha_1, \dots, \alpha_n, \alpha_i) \in J$ (as required), or $\alpha_i = \beta_i$ and $(\beta_1, \dots, \beta_n, \alpha_i) \in J$. In the latter case, by π_4 , also $(\alpha_1, \dots, \alpha_n, \alpha_i) \in J$. A similar argument applies for $\alpha_i \geq \gamma_i$.

For E1 and E3 the statement is simple consequence of condition π_4 .

This completes the proof of (i). Now, (ii) follows from Lemma 4.6, while (iii) is immediate from Theorem 3.12 and the definition of $\Gamma_+ \setminus J$.

For the converse, the existence of π has been already shown. In order to show that it is unique, suppose that $E(J, m, r) \cup \pi = E(J, m, r) \cup \pi'$. If $\pi \neq \pi'$, then they must differ in some nontrivial identity (\mathbf{a}, \mathbf{b}) , since trivial ones in both π and π' are the same. Hence, $\mathbf{a}, \mathbf{b} \notin J$, and consequently, $(\mathbf{a}, \mathbf{b}) \notin E(J, m, r)$, which contradicts the equality assumed.

4.9. Two main questions naturally arise. The first is how to find the $E(J, m, r, \pi)$ form for a theory given by a generating set of identities; and

the second, what the join and the meet of two theories given in $E(J, m, r, \pi)$ forms are. We answer these questions in the two next sections, respectively.

We conclude this section with some simple observations showing that, although remainders come in a great variety, they are in a sense finite, and for each theory can be described just by listing.

4.10. Given $m \geq 0$, $r > 0$, and $\emptyset \neq J \subseteq [(m)]$, let $\mathcal{L}(J, m, r)$ denote the set of all remainders of theories in $\mathcal{S}_S(J, m, r)$. It is not difficult to check that $\mathcal{L}(J, m, r)$ is closed under meets and joins of equivalence relations. Hence, in view of Theorem 4.8, we have the following.

4.11. **Corollary.** $\mathcal{L}(J, m, r)$ is a sublattice of the lattice of partitions of the set $\Gamma_+ \setminus J$, and $\mathcal{S}_S(J, m, r) \cong \mathcal{L}(J, m, r)$.

4.12. The least equivalence relation θ on the set $\Gamma_+ \setminus J$ (i.e., that consisting of the pairs (\mathbf{a}, \mathbf{a})) satisfies trivially conditions $\pi 0$ – $\pi 4$. Hence, the least element in $\mathcal{S}_S(J, m, r)$ is $E(J, m, r) = E(J, m, r, \theta)$, or in other words, $E(J, m, r, \pi)$ is an S -theory if and only if $\pi = \theta$. Such remainders will be referred to as *trivial*.

On the other hand, let $\rho = \rho(J, m, r)$ denote the set of all pairs $(\mathbf{a}, \mathbf{b}) \in (\Gamma_+ \setminus J)^2$ such that conditions $\pi 0$, $\pi 1$, and $\pi 3$ are satisfied, and in addition, for all $f \in \mathcal{O}$, $f(\mathbf{a}) \in J$ if and only if $f(\mathbf{b}) \in J$. It is not difficult to see that this is the greatest element in $\mathcal{L}(J, m, r)$, and therefore for the greatest element $G(J, m, r)$ in $\mathcal{S}_S(J, m, r)$ we have

4.13. **Corollary.** A nontrivial $e \in G(J, m, r)$ if and only if conditions $\pi 1$ and $\pi 3$ are satisfied, and in addition, for all $f \in \mathcal{O}$, $f(\mathbf{a}) \in J$ if and only if $f(\mathbf{b}) \in J$.

This means that every block in the partition ρ , and in consequence, in every partition $\pi \subseteq \rho$, is an antichain in (Γ, \leq) all of whose elements have the same length and are in a sense equally distant from the filter J . Moreover, for every n there is only a finite number of identities $(\alpha_1 \cdots \alpha_n, \beta_1 \cdots \beta_n) \in \pi$ (since $\alpha_i, \beta_i \leq k(J)$).

We have also the following.

4.14. **Proposition.** If $E = E(J, m, r, \pi)$, then there is a finite set $\pi_0 \subseteq \pi$ such that $e \in \pi$ if and only if there is $(\alpha_1 \cdots \alpha_n, \beta_1 \cdots \beta_n) \in \pi_0$ such that e is of the form (is E0-equivalent to)

$$\left(\begin{array}{c} \alpha_1 \cdots \alpha_n \gamma_1 \cdots \gamma_s \\ \beta_1 \cdots \beta_n \gamma_1 \cdots \gamma_s \end{array} \right)$$

for some $\gamma_1, \dots, \gamma_s > 0$ and $s \geq 0$, and $e \notin E(J, m, r)$. Moreover, $\gamma_j < \min(\alpha_i, \beta_i)$ for all j and i .

We prove this in the next section, since before we need a deeper insight into the structure of the proofs in theories of commutative semigroups.

In more detail, the remainders are considered in §6.

5. WORD PROBLEMS

A natural way to decide whether an identity e is a consequence of set of identities E is to find the form $E^* = E(J, m, r, \pi)$ given in Theorem 4.8. To

describe J it is enough to find its minimal elements (which up to equivalence, in view of Proposition 1.5 and Lemma 3.7, are finite in number). To describe π it is enough, as we are going to show, to list a finite number of identities.

5.1. We shall need two additional rules of deduction, which trivially follow from E1 and E3.

(E3⁺) replacing $\binom{\alpha_i}{\beta_i}$ by $\binom{\alpha_i+1}{\beta_i+1}$, for some i .

(E1[×]) adjoining $\binom{\gamma}{\gamma}$ column for some $\gamma \geq 1$.

It will be also convenient in the sequel to treat E0-equivalent identities as identical to avoid considering applications of the rule E0 in formal proofs. Every closure below is assumed to contain all suitable trivial identities, that is, those of the form (\mathbf{a}, \mathbf{a}) .

Now, the proof of e from E will be called *normalized* (or *of type* (E2, E3, E3⁺, E1[×], E4)), if it can be divided into five consecutive parts such that every identity in the proof, which is not already in E , is deduced by E2, whenever it occurs in the first part; by E3, whenever it occurs in the second part; by E3⁺, whenever it occurs in the third part; by E1[×], whenever it occurs in the fourth part; and by E4, whenever it occurs in the last part. In other words, first in the proof only the rule E2 is used, then only the rule E3, etc.

5.2. **Lemma.** *If $e \in E^*$, then there is a normalized proof of e from E .*

Proof. First we prove that there is a proof of type (E2, {E3, E1}, E4) (i.e., using first only the rule E2, then the rules E3 and E1, and then E4).

To this end, let C_2 be the closure of E under E2, and C_3 the closure of C_2 under E3 and E1. Then, C_3 consists of all matrices "greater or equal" than those in C_2 in a sense similar to that in Lemma 3.6, that is, every column is of the form $\binom{a}{b}$ with $a = k + \sum_{i \in I} \alpha_i$, and $b = k + \sum_{i \in I} \beta_i$, where I is not excluded to be empty. In particular, C_3 is clearly closed under E2, as well.

Now, let C_4 be the closure of C_3 under E4. Then, what we need to show is that C_4 is a closed set.

Obviously, C_4 is closed under E4. We show that it is closed under E1. Suppose that $(\mathbf{a}, \mathbf{b}) \in C_4$, and let $(\mathbf{a}1, \mathbf{b}1)$ denote here the result of the application of E1 to (\mathbf{a}, \mathbf{b}) . Then since C_4 , as a relation, is simply the transitive closure of C_3 , there is a sequence $(\mathbf{a}_0, \mathbf{a}_1), (\mathbf{a}_1, \mathbf{a}_2), \dots, (\mathbf{a}_{n-1}, \mathbf{a}_n)$ in C_3 such that $\mathbf{a}_0 = \mathbf{a}$ and $\mathbf{a}_n = \mathbf{b}$. And since C_3 is closed under E1, $(\mathbf{a}_i1, \mathbf{a}_{i+1}1) \in C_3$ for all i . In consequence, $(\mathbf{a}1, \mathbf{b}1) \in C_4$, as required. Similarly one shows that C_4 is closed under E3 and E2 (using the fact that C_3 is so).

To complete the proof it is enough to observe that every matrix in C_3 being greater or equal to a matrix in C_2 (in the sense indicated above) can be obtained from the latter by applying first the rule E3, then the rule E3⁺, and only then, if necessary, the rule E1[×].

5.3. Now, suppose we are given a finite set of identities E . Then, according to Theorem 4.8, $E^* = E(J, m, r, \pi)$ for some J, m, r, π . In §5.5 we present an algorithm to find this form. Before, we describe it in a less formal way, introducing some special notions we need, and making suitable comments. An example of application is given in §5.8.

5.4. In the first step of the algorithm, using Lemma 2.4, we find the numbers m, r , and $k = k(J)$. Since, $(k) \in J$, we have $E^* = (E_1 \cup E(J_1, m, r))^*$,

where $J_1 = [(k)]$ is the filter generated by the sequence (k) , and E_1 is the set obtained from E by deleting the identities with α_i or $\beta_i \geq k$ for some i . In the sequel, we call such identities (k) -identities.

Further, in every step of algorithm, we form a new set E_i of identities by deleting some identities from E_{i-1} , and a new filter J_i by *adjoining* some sequences to J_{i-1} in such a way that the equation

$$(13) \quad E^* = (E_i \cup E(J_i, m, r))^*$$

still holds. (By *adjoining* sequences to a filter J we mean forming a new filter generated by J and the adjoined sequences.) Our ultimate goal is to obtain the filter $J_s = J$ and a finite set $E_s \subseteq \pi$ determining π in a straightforward way.

To this end, in the second step, we adjoin to the filter J_1 all the sequences corresponding to S -identities and irregular identities in E_1 , and then, delete all these identities from E_1 . The resulting pair we denote by (J_2, E_2) . In view of remarks in §3.16, equation (13) holds.

Now, we “separate” J_2 and E_2 in the following way. If there is an identity $(a, b) \in E_2$ such that either a or $b \in J_2$, then we delete (a, b) from E_2 , and adjoin b or a , respectively, to J_2 . In view of Theorem 4.8(iii), this preserves equation (13). We repeat this until we obtain the filter J_3 and the set of identities E_3 such that $(a, b) \in E_3$ implies $a, b \notin J_3$.

To make our algorithm more efficient, we apply this procedure (whose input and output are pairs (J, E)) also in further steps, and it will be referred to as the *separating subroutine* (in short, SS).

For the third step, note that in view of condition $\pi 3$, all the sequences $(\alpha_1, \dots, \alpha_n, \min(\alpha_i, \beta_i))$, and $(\beta_1, \dots, \beta_n, \min(\alpha_i, \beta_i))$, for all $(a, b) \in E$, and all i with $\alpha_i \neq \beta_i$, are in J . Hence, our third step is to adjoin all these sequences to J_3 ; they will be referred to as the *sequences given by $\pi 3$* . Then, we apply the separating subroutine. The resulting pair we denote by (J_4, E_4) .

Now, our idea is to apply Lemma 5.2 to determine all the consequences of the set $E_4 \cup E(J_4, m, r)$. First, observe that this set is already closed under E_2 (since $E(J_4, m, r)$ itself is a theory, and every E_2 -consequence of E_4 , in view of the third step, is in $E(J_4, m, r)$). Hence, as the next step, we wish to find the closure under E_3 . Of course, it is enough to consider only E_3 -consequences of the set E_4 (since $E(J_4, m, r)$ is closed), and only those, which are not already in $E(J_4, m, r)$. Such consequences are finite in number (because all the sets E_i are finite, and there is only a finite number of sequences of a fixed length not in J). We adjoin them to E_4 , and after applying the separating subroutine, obtain (J_5, E_5) .

The fifth step is analogous, but with E_3^+ used instead of E_3 . The sixth step is also analogous, except that to keep the set E_i finite, we adjoin only E_1^{\times} -consequences in no more than s variables, where the number $s = s(E)$ is defined to be the maximum of $d(J_4)$ (defined in §3.14) and the length $n(E_4)$ of the longest sequence a in some nontrivial $(a, b) \in E_4$. The resulting pair we denote by (J_7, E_7) .

Now, we take the closure of E_7 under E_4 (i.e., the transitive closure of E_7). It may happen that new S -identities appear. We replace these by the corresponding sequences adjoined to J_7 , and to close, apply once again the separating subroutine. Then, the resulting set $E_8 \cup E(J_8, m, r)$ is closed under E_4 , and no S -identity, and no identity (a, b) with a or b in J is in E_8 .

Clearly, $E^* = (E(J_8, m, r) \cup E_8^{*\times})^{*4}$, where $^{*\times}$ denotes the closure under $E1^\times$, and *4 , the closure under E4. Since $E_8^{*\times}$ contains no S -identity, all we need to show is that *4 is superfluous, that is, that $E(J_8, m, r) \cup E_8^{*\times}$ is closed under E4.

Before, however, we put the result in a more formal way.

5.5. Algorithm to find the $E(J, m, r, \pi)$ -form and solve word problems.

input E

1. Find k, m and r , using Lemma 2.4.
 2. Form the filter J generated by the sequence (k) and all the sequences corresponding to S -identities and irregular identities.
Delete these identities and (k) -identities from E .
Apply SS (the separating subroutine) to (J, E) .
 3. Adjoin to J all sequences given by π_3 , and apply SS .
 4. Complete E with all E3-consequences.
Replace S -identities in E by the corresponding sequences adjoined to J , and apply SS .
 5. Complete E with all E3⁺-consequences and apply SS .
 6. Put $s = \max(d(J), n(E))$.
Complete E with all $E1^\times$ -consequences in no more than s variables, and apply SS .
 7. Form the transitive closure of E .
Replace S -identities in E by the corresponding sequences adjoined to J , and apply SS .
Put $\pi(E) = E$.
- output** $J, m, r, \pi(E)$

The main result of this section is

5.6. Theorem. *Given a finite set of identities E , if $J, m, r, \pi(E)$ is the output of the algorithm 5.5, and $^{*\times}$ denotes the closure under the rule $E1^\times$, then*

$$E^* = E(J, m, r, \pi), \quad \text{where } \pi = \pi(E)^{*\times} \setminus J^2.$$

Consequently, $e = (\mathbf{a}, \mathbf{b}) \in E^*$ if and only if either e satisfies conditions N3–N4 and both $\mathbf{a}, \mathbf{b} \in J$ or there is $(\alpha_1 \cdots \alpha_n, \beta_1 \cdots \beta_n) \in \pi(E)$ such that e is of the form (is E0-equivalent to)

$$\begin{pmatrix} \alpha_1 \cdots \alpha_n \gamma_1 \cdots \gamma_t \\ \beta_1 \cdots \beta_n \gamma_1 \cdots \gamma_t \end{pmatrix}$$

for some $\gamma_1, \dots, \gamma_t > 0$ and $t \geq 0$.

Proof. In view of our description in §5.4, and Theorem 3.12, we have only to prove that $E(J, m, r) \cup \pi(E)^{*\times}$ is closed under E4. First we make two observations.

(a) If $e \in \pi(E)^{*\times} \setminus J^2$ and $\alpha_1 \geq \cdots \geq \alpha_n$, then $e_1 = (\alpha_1 \cdots \alpha_s, \beta_1 \cdots \beta_s) \in \pi(E)$, and $\alpha_i = \beta_i$ for all $i > s$. Indeed, if e is an $E1^\times$ -consequence of some $e_2 = (\alpha_{i_1} \cdots \alpha_{i_s}, \beta_{i_1} \cdots \beta_{i_s}) \in \pi(E)$, then in view of Step 3, all $\alpha_{i_j}, \beta_{i_j}$ in e_2 are greater than every other α_j or β_j in e (otherwise, \mathbf{a} or \mathbf{b} belong to J). Whence, $\{i_1, \dots, i_s\} = \{1, \dots, s\}$.

(b) If $e \in \pi(E)^{*\times}$, then $(\alpha_1, \dots, \alpha_n) \in J$ if and only if $(\beta_1, \dots, \beta_n) \in J$. Indeed, suppose that $\alpha_1 \geq \cdots \geq \alpha_n$ and $(\alpha_1, \dots, \alpha_n) \in J$. Then, since

$s \geq d(J)$, also $(\alpha_1, \dots, \alpha_s) \in J$. Consequently, in view of remarks at the end of §5.4, $(\beta_1, \dots, \beta_s) \in J$; and it follows that also $(\beta_1, \dots, \beta_n) \in J$, as required.

Now, from (b), and since $E(J, m, r)$ is closed, we see that in order to show that $E(J, m, r) \cup \pi(E)^{\ast\ast}$ is closed under E4, it is enough to show that if $e_1 = (\alpha_1 \cdots \alpha_n, \beta_1 \cdots \beta_n)$ and $e_2 = (\beta_1 \cdots \beta_n, \gamma_1 \cdots \gamma_n)$ are in $\pi(E)^{\ast\ast} \setminus J^2$, then $e_3 = (\alpha_1 \cdots \alpha_n, \gamma_1 \cdots \gamma_n) \in \pi(E)^{\ast\ast}$.

In §5.4, we observed that $\pi(E)$ itself is closed under E4. In view of (a), assuming that $\beta_1 \geq \cdots \geq \beta_n$, both $(\alpha_1 \cdots \alpha_s, \beta_1 \cdots \beta_s), (\beta_1 \cdots \beta_s, \gamma_1 \cdots \gamma_s) \in \pi(E)$, and $\alpha_i = \gamma_i$ for $i > s$. Consequently, $(\alpha_1 \cdots \alpha_s, \gamma_1 \cdots \gamma_s) \in \pi(E)$, and $(\alpha_1 \cdots \alpha_n, \gamma_1 \cdots \gamma_n) \in \pi(E)^{\ast\ast}$, which completes the proof.

5.7. Since by Proposition 1.2 every theory is finitely generated, we see now that the result announced in Proposition 4.14 is an immediate consequence of Theorem 5.6 and the observation (a) in the proof.

A natural equational base corresponding to the result in Theorem 5.6 is that consisting of the identities in $B(J, m, r)$ and the identities in $\pi(E)$ (which can be usually reduced so that the hypothesis of Theorem 5.6 still holds). One can observe, that if E involves no more than k variables, then the base in question involves no more than $k + 1$ variables.

5.8. As an example, let E consist of the following identities

$$x^4 y^2 z^3 = x^2 y^6 z^5, \quad x y z^3 = x^3 y z, \quad x^7 y^3 z^3 t^6 = x^9 y^5 z^1 t^2.$$

Then, as the first step, we find $k = 6$, $m = 1$, and $r = 2$. There is no S -identity in E . Therefore, at the second step, $J = [(6)]$, and $E = \{(\begin{smallmatrix} 423 \\ 265 \end{smallmatrix}), (\begin{smallmatrix} 113 \\ 311 \end{smallmatrix})\}$. The sequence $(265) \in J$; whence, applying SS we get $J = [(6), (423)]$ and $E = \{(\begin{smallmatrix} 113 \\ 311 \end{smallmatrix})\}$.

In Step 3, we get $J = [(6), (423), (3111), \dots] = [(3111)]$, and E without change. In Step 4, $E = \{(\begin{smallmatrix} 113 \\ 311 \end{smallmatrix}), (\begin{smallmatrix} 23 \\ 41 \end{smallmatrix})\}$.

For Step 5, note that all nontrivial E3⁺-consequences of E are in $E(J, 1, 2)$; whence, nothing changes in this step. For Step 6 the situation is the same (we note only that $s = \max(4, 3) = 4$). And since E is closed under E4, we have as a final result $J = [(3111)]$, and $\pi(E) = \{(\begin{smallmatrix} 113 \\ 311 \end{smallmatrix}), (\begin{smallmatrix} 23 \\ 41 \end{smallmatrix})\}$.

In this particular case, there are only two nontrivial identities in the remainder; namely, those listed in $\pi(E)$. All other nontrivial identities belong to $E(J, 1, 2)$.

5.9. Consider another example. When is the identity

$$(14) \quad x_1^{\beta_1} x_2^{\beta_2} \cdots x_s^{\beta_s} = x_1^{\beta_1+1} x_2^{\beta_2} \cdots x_s^{\beta_s}$$

($n \geq 1$) a consequence of the identity

$$(15) \quad x_1^{\alpha_1} \cdots x_n^{\alpha_n} = x_0 x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

($s \geq 1$) in the theory of commutative semigroups?

Clearly, (15) is an S -identity, and therefore, for the theory generated by this identity, $J = [(\alpha_1, \dots, \alpha_n)]$, $m = 0$, $r = 1$, and $\pi(E) = \emptyset$. Conditions N3–N4 are in this case always satisfied. Hence, in view of Theorem 5.6, (14) is a consequence of (15) if and only if the sequence $(\beta_1, \dots, \beta_s) \in J$, that is, if and only if

$$(16) \quad (\alpha_1, \dots, \alpha_n) \leq (\beta_1, \dots, \beta_s).$$

Now observe, that in view of Lemma 3.6, this condition holds if and only if n items of the sizes $\alpha_1, \dots, \alpha_n$, respectively, can be packed into s bins of the sizes β_1, \dots, β_s . This is the so-called BIN-PACKING problem, which is well known to be *NP*-complete (see, e.g., [10]). In consequence, we have the following.

5.10. Proposition. *The problem to decide whether an identity is a consequence of a finite set of commutative semigroup identities is NP-hard.*

For related results see [23, 38].

6. LATTICE \mathcal{L}_{Com}

6.1. In order to describe the structure of the whole lattice \mathcal{L}_{Com} (and thus the structure of the lattice of varieties $\mathcal{L}(\text{Com})$) we have to find the join and meet of $E_1 = E(J_1, m_1, r_1, \pi_1)$ and $E_2 = E(J_2, m_2, r_2, \pi_2)$. The main problem is that suggested already in Lemma 4.3. Namely, it may happen that there is a nontrivial $(\mathbf{a}, \mathbf{b}) \in \pi_1$ such that $\mathbf{a} \in J_2$, while $\mathbf{b} \notin J_2$. Then, $(\mathbf{a}, \mathbf{b}) \in E_1 \vee E_2 = E(J_3, m_3, r_3, \pi_3)$, and since $\mathbf{a} \in J_1 \cup J_2 \subseteq J_3$ (cf. Lemma 4.3), it has to be, in view of Theorem 4.8(iii), $\mathbf{b} \in J_3 \setminus J_1 \cup J_2$. Hence, $J_3 \neq J_1 \cup J_2$.

For E_1, E_2 as above, we define J_u to be the least filter $J \supseteq J_1 \cup J_2$ such that whenever $(\mathbf{a}, \mathbf{b}) \in \pi_1 \cup \pi_2$ and either \mathbf{a} or \mathbf{b} is in $J_1 \cup J_2$, then both $\mathbf{a}, \mathbf{b} \in J$. Clearly, $J_3 \supseteq J_u$, and we are going to show that actually the equality holds.

It is useful, however, to have a more direct definition of J_u . Given a set J and an equivalence relation π , let $C(J, \pi)$ be the union of all those blocks in π , which have a point in common with J , completed with all equivalent sequences with zeros (i.e., $\mathbf{a} \in C(J, \pi)$ if for the sequence \mathbf{a}_0 obtained from \mathbf{a} by deleting all zeros there is $\mathbf{b} \in J$ such that $(\mathbf{a}, \mathbf{b}) \in \pi$). Then, it is easy to observe, that $J_u = J_1 \cup J_2 \cup C$, where $C = C(J_1 \cup J_2, \pi_1 \vee \pi_2)$, and \vee denotes the join of equivalence relations.

Below, we use E^S to denote the Schwabauer part of the theory E (i.e., for $E = E(J, m, r, \pi)$, $E^S = E(J, m, r)$), and with superscripts we denote cartesian powers. Then, we have the following.

6.2. Theorem. *If $E_1 = E(J_1, m_1, r_1, \pi_1)$, and $E_2 = E(J_2, m_2, r_2, \pi_2) \in \mathcal{L}_{\text{Com}}$, then the following hold.*

- (i) $E_1 \supseteq E_2$ if and only if $J_1 \supseteq J_2$, $m_1 \leq m_2$, r_1 divides r_2 , and $\pi_1 \cup J_1^2 \supseteq \pi_2$.
- (ii) $E_1 \vee E_2 = E(J_1 \cup J_2 \cup C, \min(m_1, m_2), \gcd(r_1, r_2), \pi_1 \vee \pi_2 \setminus C^2)$, where $C = C(J_1 \cup J_2, \pi_1 \vee \pi_2)$.
- (iii) $E_1 \cap E_2 = E(J_1 \cap J_2, \max(m_1, m_2), \text{lcm}(r_1, r_2), \pi)$, where $\pi = (\pi_1 \cap \pi_2) \cup (\pi_1 \cap E_2^S) \cup (\pi_2 \cap E_1^S)$.

Moreover, if $E_i = E(J_i, m_i, r_i, \pi_i)$ is a family of theories in \mathcal{L}_{Com} , then

- (iv) $\bigvee E_i = E(\bigcup J_i \cup C, \min\{m_i\}, \gcd\{r_i\}, \bigvee \pi_i \setminus C^2)$, where $C = C(\bigcup J_i, \bigvee \pi_i)$.
- (v) $\bigcap E_i = E(\bigcap J_i, \max\{m_i\}, \text{lcm}\{r_i\}, \bigcap (\pi_i \cup E_i^S) \setminus \bigcap E_i^S)$, provided $\bigcap J_i$ is nonempty and both $\max\{m_i\}$ and $\text{lcm}\{r_i\}$ exist; if not, $\bigcap E_i = E_\emptyset$.

Proof. We use Theorem 3.12. For (i), the “only if” part is trivial.

For the “if” part it is enough to show that if $(\mathbf{a}, \mathbf{b}) \in \pi_2 \subseteq \pi_1 \cup J_1^2$, then $(\mathbf{a}, \mathbf{b}) \in E_1$. If $(\mathbf{a}, \mathbf{b}) \in \pi_1$, then the result is by the definition of E_1 . If $(\mathbf{a}, \mathbf{b}) \in J_1^2$, then since it satisfies condition π_1 , $(\mathbf{a}, \mathbf{b}) \in E(J_1, m_1, r_1) \subseteq E_1$, as required.

For (ii), let us denote $J = J_1 \cup J_2 \cup C$. First we show that

$$E_3 = E(J, \min(m_1, m_2), \gcd(r_1, r_2), \pi)$$

with $\pi = \pi_1 \vee \pi_2 \setminus C^2$ is a theory, that is, satisfies the conditions of Theorem 4.8. Since $E_1 \in \mathcal{S}_N(k(J_1), m_1, r_1)$, $E_2 \in \mathcal{S}_N(k(J_2), m_2, r_2)$, we have, by virtue of Lemma 2.4, $m(J) \geq \min(m_1, m_2)$. Hence, it is enough to check that $\pi = \pi_1 \vee \pi_2 \setminus C^2$ is an equivalence relation on the set $\Gamma_+ \setminus J$ satisfying conditions π_0 – π_4 . Of these, π_0 – π_3 are easy to check. We check π_4 .

First note that if $(\mathbf{a}, \mathbf{b}) \in \pi_1 \cup \pi_2$ and for some $f \in \mathcal{O}$ one of $f(\mathbf{a})$ or $f(\mathbf{b})$ is in J , then both $f(\mathbf{a}), f(\mathbf{b}) \in J$. This is immediate, if (according to π_4) both $f(\mathbf{a}), f(\mathbf{b}) \in J_1$ or J_2 , and follows from the definition of J , if $(f(\mathbf{a}), f(\mathbf{b})) \in \pi_1 \cup \pi_2$.

Now, if $(\mathbf{a}, \mathbf{b}) \in \pi_1 \vee \pi_2 \setminus C^2$, then there is a sequence

$$(\mathbf{a}, \mathbf{a}_1), (\mathbf{a}_1, \mathbf{a}_2), \dots, (\mathbf{a}_n, \mathbf{b})$$

such that each element is in π_1 or π_2 . Whence, if $f(\mathbf{a}_i) \in J$ for at least one i , then by the remark above both $f(\mathbf{a}), f(\mathbf{b}) \in J$, as required. Otherwise, by π_4 , all elements in the sequence $(f(\mathbf{a}), f(\mathbf{a}_1)), (f(\mathbf{a}_1), f(\mathbf{a}_2)), \dots, (f(\mathbf{a}_n), f(\mathbf{b}))$ are in π_1 or in π_2 . Consequently, $f(\mathbf{a}), f(\mathbf{b}) \in \pi_1 \vee \pi_2 \setminus C^2$, proving the property π_4 , and thus showing that E_3 is a theory.

Now, by (i), $E_3 \supseteq E_1, E_2$, and consequently, $E_3 \supseteq E_1 \cup E_2$. On the other hand, by the argument in §6.1,

$$E(J, \min(m_1, m_2), \gcd(r_1, r_2)) \subseteq E_1 \vee E_2,$$

and obviously, $\pi_1 \vee \pi_2 \subseteq E_1 \vee E_2$. This completes the proof of (ii).

For (iii), compute

$$E_1 \cap E_2 = (E_1^S \cup \pi_1) \cap (E_2^S \cup \pi_2) = (E_1^S \cap E_2^S) \cup (E_1^S \cap \pi_2) \cup (E_2^S \cap \pi_1) \cup (\pi_1 \cap \pi_2).$$

Hence, by Theorems 3.12 and 4.8,

$$E_1 \cap E_2 = E(J_1 \cup J_2, \max(m_1, m_2), \text{lcm}(r_1, r_2), (E_1^S \cap \pi_2) \cup (E_2^S \cap \pi_1) \cup (\pi_1 \cap \pi_2)).$$

By Lemma 4.3 and in view of the description of the equivalence relation π in §4.7, this is the required form $E(J, m, r) \cup \pi$.

The proofs of (iv) and (v) are analogous and are left to the reader.

6.3. For applications, it is useful to note the following. If $J_1 \supseteq J_2$, then formulations in (ii) and (iii) simplify to

$$(ii)^* \quad E_1 \vee E_2 = E(J_1, \min(m_1, m_2), \gcd(r_1, r_2), \pi_1 \vee \pi_2 \setminus J_1^2),$$

$$(iii)^* \quad E_1 \cap E_2 = E(J_2, \max(m_1, m_2), \text{lcm}(r_1, r_2), \pi_2 \cap (\pi_1 \cup E_1^S)).$$

6.4. Using the formulas above for $E = E(J, m, r, \pi)$ and $k = k(J)$, we obtain the following identity.

$$(17) \quad E = (E \vee E(k, 0, 1)) \cap E(1, 0, r) \cap E(m, m, 1).$$

Apply it to varieties. If $\text{Nil}_k = \mathcal{V}(k, 0, 1)$ is the variety of nil-semigroups defined by $x^k = 0$, $\mathcal{A}_r = \mathcal{V}(1, 0, r)$ is the variety of abelian groups defined

by $x^r = 1$, and $\text{Ap}_m = \mathcal{V}(m, m, 1)$ is the variety of aperiodic semigroups defined by $x^{m+1} = x^m$, then, for a variety $\mathcal{V} = \mathcal{V}(J, m, r, \pi)$ corresponding to the equational theory $E = E(J, m, r, \pi)$, we have

$$(18) \quad \mathcal{V} = (\mathcal{V} \cap \text{Nil}_k) \vee \mathcal{A}_r \vee \text{Ap}_m,$$

a result proved in [24].

6.5. Another important result from [24] also easily follows. The mapping

$$(19) \quad \phi : \mathcal{V}(J, m, r, \pi) \rightarrow (\mathcal{V}(J, 0, 1, \pi), m, r)$$

of $\mathcal{L}(\text{Com})$ into the cartesian product of $\mathcal{L}(\text{Nil}) \times N_0 \times N_1$ (where N_0 and N_1 are the natural numbers with and without zero, ordered naturally and by division, respectively) is an embedding of meet-semilattices. This is straightforward from Theorem 6.2(ii).

In contrast to [24], however, one can see explicitly why joins (resp., equational theory meets) are not preserved (π in (iii) depends on m_i and r_i), and an explicit counterexample can be easily given.

Let $\bar{\phi} : E(J, m, r, \pi) \rightarrow (E(J, 0, 1, \pi), m, r)$ be the map dual to ϕ . Let $E_1 = E(1, 0, 2)$, and $E_2 = (J_{(211)}, 0, 1, \pi)$, where $J_{(211)}$ is the filter generated by the sequence $(2, 1, 1)$, and π is the remainder generated by the identity $x^2y = xy^2$. Then, by (iii)*, $E_1 \cap E_2 = E(J_{(211)}, 0, 2)$ (since $x^2y = xy^2 \notin E_1$), while $\bar{\phi}(E_1) = (E(1, 0, 1), 0, 2)$, $\bar{\phi}(E_2) = (E_2, 0, 1)$, and $\bar{\phi}(E_1) \cap \bar{\phi}(E_2) = (E_2, 0, 2) \neq \bar{\phi}(E_1 \cap E_2)$.

We proceed to describe the greatest element in the interval $\mathcal{I}_S(J, m, r)$. This will shed more light on the structure of the remainders.

6.6. **Lemma.** *Let $E = E(J, m, r, \pi)$. If $e \in \pi$ is of the form (1) and $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n$, then $\alpha_i = \beta_i$ for all $i \geq d(J)$.*

Proof. Suppose that for some $j \geq d(J)$, $\alpha_j < \beta_j$. Then, by $\pi 3$, the sequence $(\alpha_1, \dots, \alpha_j, \alpha_j, \alpha_{j+1}, \dots, \alpha_n) \in J$. It follows that there is \mathbf{a} in the fundamental antichain of J such that $\mathbf{a} \leq [\alpha_1, \dots, \alpha_j, \alpha_j, \alpha_{j+1}, \dots, \alpha_n]$. By assumption and since the length of \mathbf{a} is less or equal to $d(J) \leq j$, it follows in view of Lemma 3.6, that $\mathbf{a} \leq [\alpha_1, \dots, \alpha_j]$. Consequently, $(\alpha_1, \dots, \alpha_n) \in J$, a contradiction.

Note that the result in Lemma 6.6 does *not* mean that π is determined by the identities in at most $n_0 = d(J)$ variables. Generally, an integer n_0 with this property always exists, but may be arbitrary large and does not depend on J at all. In the special case of ρ we have the following.

6.7. **Proposition.** *Let $G(J, m, r) = E(J, m, r, \rho)$, $e \in \rho$, and assume that the least number among all α_i and β_i is α_n . Denote $n_0 = (k(J) - 1)(d(J) - 1)$. If e is nontrivial and $n > n_0$, then $(\alpha_1 \dots \alpha_{n-1}, \beta_1 \dots \beta_{n-1}) \in \rho$ and $\alpha_n = \beta_n$.*

In particular, $G(J, m, r)$ has a basis involving no more than n_0 variables.

Proof. Denote $(\mathbf{a}, \mathbf{b}) = (\alpha_1 \dots \alpha_{n-1}, \beta_1 \dots \beta_{n-1})$. By Lemma 6.6, since e is nontrivial, $k(J), d(J) > 1$ and $\alpha_n = \beta_n > 0$.

By virtue of Corollary 4.13, (\mathbf{a}, \mathbf{b}) satisfies condition $\pi 1$ (for N1 and N2 note that $n \geq k(J)$). An argument similar to that in the proof of Lemma 6.6 shows that (\mathbf{a}, \mathbf{b}) also satisfies $\pi 3$.

For the remaining condition suppose, on the contrary, that $f(\mathbf{a}) \in J$ and $f(\mathbf{b}) \notin J$ for some $f \in \mathcal{O}$, and denote by $f(\mathbf{b})\beta_n$ the sequence obtained from

$f(\mathbf{b})$ by adjoining β_n . Then, since $f(\mathbf{a}) \in J$, $f(\mathbf{a})\alpha_n \in J$, and since $e \in \rho$, also $f(\mathbf{b})\beta_n \in J$. Since $\beta_n = \alpha_n$ is minimal, it follows that the length of $f(\mathbf{b})$ is less than $d(J)$ (otherwise, similarly as in the proof of Lemma 6.6, $f(\mathbf{b}) \in J$). Since $\sum f(\mathbf{b}) > n_0$, by the pigeonhole principle one of the elements of the sequence $f(\mathbf{b})$ must be not less than $k(J)$. Whence, $f(\mathbf{b}) \in J$, a contradiction.

Now, let G be the set of all identities in $\rho(J, m, r)$ in at most n_0 variables. In view of the remarks following Corollary 4.13, G is finite. The result in Proposition 6.7 means that every identity in ρ , and therefore every identity in $\pi \subseteq \rho$, is an $E1^\times$ -consequence of some member of G . In particular, in view of Proposition 4.14, every π is $E1^\times$ -generated by a finite number of $E1^\times$ -consequences of G . Although this representation is not unique, in many cases it helps to determine completely the interval $\mathcal{I}_S(J, m, r)$.

We conclude this section with a number of examples establishing some simple facts and illustrating how varied the remainders and the intervals $\mathcal{I}_S(J, m, r)$ can be.

Example 1. Let J be any filter generated by sequences of the length at most 2. Then, by Lemma 6.6, for every $E = E(J, m, r, \pi)$, π is trivial, and consequently, $\mathcal{I}_S(J, m, r) = \{E(J, m, r)\}$ has precisely one element.

Example 2. Suppose that $(m, 1, 1, \dots, 1) \in J$, where $m > 0$, and 1 occurs $s > 0$ times. Then, every identity e with $n > s$ satisfying N1–N4 is in $E(J, m, r)$. Whence, $\rho = \rho(J, m, r)$ consists only of a finite number of nontrivial identities, and consequently, $\mathcal{I}_S(J, m, r)$ is finite, as well. Obviously, for every J with this property, $\mathcal{I}_S(J, m, r)$ can be effectively described. The problem with a uniform description of these intervals is that they are as varied as possible. In fact, every finite lattice can be embedded into one of these intervals, as the next example shows.

Example 3. Let $J = [(1, 1, \dots, 1)]$ with 1 occurring $s + 1$ times and, to simplify the example, let $m = r = 1$. Then, by Corollary 4.13, in particular, every identity of the form $x^u y^{s-u} = x^w y^{s-w}$ ($0 < u, w < s$) is in $\rho(J, m, r)$. Note, that every nontrivial consequence of any of these identities is in $E(J, m, r)$; so they are independent. Now, to every partition of the set of terms $\{x^1 y^{s-1}, x^2 y^{s-2}, \dots, x^t y^{s-t}\}$ (with $t \leq s/2$) we assign the theory $E(J, m, r, \pi)$, where π is generated by identities corresponding to the pairs in the blocks of the partition. Clearly, such theories form a sublattice of the lattice $\mathcal{I}_S(J, m, r)$ isomorphic to the partition lattice Π_t .

This is, in fact, precisely the result of [4].

Example 4. If $\rho = \rho(J, m, r)$ contains at least one nontrivial identity $(\alpha_1 \cdots \alpha_n, \beta_1 \cdots \beta_n)$ with $n > d(J)$, then $\mathcal{I}_S(J, m, r)$ is infinite. Indeed, in such a case every identity of the form (\mathbf{a}, \mathbf{b}) , where

$$\mathbf{a} = (\alpha_1, \dots, \alpha_n, 1, 1, \dots, 1) \quad \text{and} \quad \mathbf{b} = (\beta_1, \dots, \beta_n, 1, 1, \dots, 1),$$

is in ρ , and all these identities (together with the S -identities corresponding to J) generate an infinite descending chain of theories in $\mathcal{I}_S(J, m, r)$.

If, in addition, all integers occurring in the sequences belonging to the fundamental antichain of J are greater than one, then the construction above also

shows that in such a case $\mathcal{S}(J, m, r)$ has no atoms. On the other hand, by Proposition 1.2, $\mathcal{S}(J, m, r)$ has at least one coatom; and by a result of Almeida [2] mentioned in the introduction, the number of coatoms (atoms) in $\mathcal{S}(J, m, r)$ is always finite.

Example 5. Consider the sublattice $\mathcal{L}(\mathcal{N})$ of $\mathcal{L}(\text{Com})$, of varieties of nilpotent commutative semigroups. It is the union of the ascending chain of the lattices $\mathcal{L}(\mathcal{N}_k)$ of subvarieties of the variety \mathcal{N}_k of commutative semigroups defined by the law $x_1 \cdots x_k = 0$. The equational theory of \mathcal{N}_k is $E(\mathcal{N}_k) = E(J_k, 0, 1)$, where $J_k = [(1, 1, \dots, 1)]$ with 1 occurring k times.

There are only finitely many sequences $(\alpha_1, \dots, \alpha_n)$ with $\sum \alpha_i < k$, and therefore the lattices $\mathcal{L}(\mathcal{N}_k)$ are all finite. Thus, every proper interval of $\mathcal{L}(\mathcal{N}_k)$ is finite, and can be explicitly described, using Theorem 6.2 and Proposition 6.7.

Example 6. If $E = E(J, m, r, \pi)$ and a nontrivial $e \in \pi$, then by conditions $\pi 0$ and $\pi 4$, e is regular, and $(\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n)$ are noncomparable in (Γ, \leq) . It is not difficult to see that every identity e satisfying these conditions is a member of the remainder π of some theory E .

Indeed, let for example, $e : x^2 y z^5 = x^3 y^3 z$ ($\mathbf{a} = (2, 1, 5)$, $\mathbf{b} = (3, 3, 1)$). We construct a filter J such that $f(\mathbf{a}), f(\mathbf{b}) \in J$ for all nontrivial $f \in \mathcal{O}$. There is a unique such filter; namely, in this case, $J = [(2, 1, 5, 1), (3, 3, 1, 1)]$. Then, e satisfies obviously conditions $\pi 0$ – $\pi 4$, and for instance, $e \in \rho(J, 1, 1)$.

This construction works for any regular identity with noncomparable sequences of the exponents, and may be used to construct various special examples. In fact, it was used to construct the counterexample in §6.5.

7. FINITE SEMIGROUPS

In this section we apply our results to finite semigroups and finitely generated varieties. First we introduce some notation.

7.1. Given a commutative semigroup \mathbf{S} and a variety \mathcal{V} , by $E(\mathbf{S})$ and $E(\mathcal{V})$ we denote the equational theories of \mathbf{S} and \mathcal{V} , respectively. If $E(\mathbf{S}) = E(J, m, r, \pi)$, then we denote $k(\mathbf{S}) = k(J)$, $m(\mathbf{S}) = m$, and $r(\mathbf{S}) = r$. Similarly, we define $k(\mathcal{V})$, $m(\mathcal{V})$, and $r(\mathcal{V})$ for $E(\mathcal{V}) = E(J, m, r, \pi)$.

Note, that for a given finite \mathbf{S} , using the definitions in §2.2, all these parameters can be easily computed. Below, we shall see, in particular, how other parameters of $E(\mathbf{S})$ can be computed.

We start from the following.

7.2. **Lemma.** *Let $E \in \mathcal{S}_N(k, m, r)$ and let an identity*

$$(20) \quad x_1^m x_2 \cdots x_n = x_1^{m+r} x_2 \cdots x_n$$

be in E for some $n \geq 1$. Then,

- (i) *for $m > 0$, every nontrivial identity in at least n variables satisfying conditions N1–N4 is in E ,*
- (ii) *for $m = 0$, every nontrivial identity satisfying conditions N1–N4 and such that at least $n - 1$ variables have occurrences on both the sides is in E .*

Proof. If e is a nontrivial identity satisfying the conditions of (i), then both $(\alpha_1, \dots, \alpha_n), (\beta_1, \dots, \beta_n)$ are in the filter generated by $(m, 1, \dots, 1)$, with 1 occurring $n - 1$ times. Applying Theorem 3.12 yields the desired result. A similar argument proves (ii).

7.3. Proposition. *Let S be a finitely generated commutative semigroup with q generators. If $k(S) = k$ and $m(S) = m$, then the identity (20) holds in S for every $n \geq q(k - 1) - m + 2$. In consequence, conditions (i) and (ii) of Lemma 7.2 hold for $E(S)$.*

Proof. Let a_1, a_2, \dots, a_n be arbitrary elements of S . Representing each a_i as a product of generators g_i , we get

$$a_1^m a_2 \cdots a_n = g_1^{\alpha_1} \cdots g_q^{\alpha_q}$$

for some $\alpha_1, \dots, \alpha_q \geq 0$ with $\sum \alpha_i \geq m + n - 1 \geq q(k - 1) + 1$. It follows that $\alpha_i \geq k$ for some i . Whence, we may assume that

$$a_1^m a_2 \cdots a_n = g_1^{\alpha_1} \cdots g_s^{\alpha_s} g_{s+1}^{\alpha_{s+1}} \cdots g_q^{\alpha_q}$$

where $\alpha_1 \geq k$, $\alpha_i \geq m$ for $i \leq s$, and $\alpha_i < m$, otherwise ($1 \leq s \leq q$). Then, in view of (8),

$$a_1^{m+r} a_2 \cdots a_n = g_1^{\beta_1} \cdots g_s^{\beta_s} g_{s+1}^{\alpha_{s+1}} \cdots g_q^{\alpha_q},$$

where $\beta_1 \geq k$, $\beta_i \geq m$ for $i \leq s$, and $\beta_i \equiv \alpha_i \pmod{r}$, where $r = r(S)$. Hence, by (8),

$$a_1^m a_2 \cdots a_n = a_1^{m+r} a_2 \cdots a_n$$

and the result follows.

It follows from the proposition above that if $E(S) = E(J, m, r, \pi)$, then $[(m, 1, \dots, 1)] \subseteq J \subseteq [(m)]$ with 1 occurring $q(k - 1) - m + 1$ times. There are only finitely many filters with this property, and therefore, for finite S , J can be effectively computed without appeal to Theorem 5.6. To compute π the next proposition applies.

7.4. Proposition. *If a semigroup S has q generators, $k(S) = k$ and $m(S) = m$, then S has a basis for its identities involving (besides $x(yz) = (xy)z$, $xy = yx$) no more than $n = q(k - 1) - m + 2$ variables.*

Proof. If $m > 0$, then the result is a simple consequence of Proposition 7.3. Hence, suppose that $m = 0$ and let

$$(21) \quad x_1^{\alpha_1} \cdots x_s^{\alpha_s} w = z_1^{\beta_1} \cdots z_t^{\beta_t} w$$

be an identity holding in S , where w is a word, and $x_1, \dots, x_s, z_1, \dots, z_t$ are pairwise distinct variables not occurring in w . We wish to show that this is a consequence of the set $Id_n(S)$ of identities in at most n variables, holding in S .

Obviously, $r = r(S)$ divides both α_i and β_i for all i , and we may assume that $s > 0$, and $t \geq 0$ (because, if both $s = t = 0$, then (21) either is in $Id_n(S)$, or by Lemma 7.2(ii), is a consequence of $Id_n(S)$).

First we need some further identities holding in S . Substituting $x_i = y^k$ in (21), for all i , and applying (3), which is in $Id_n(S)$, we get

$$(22) \quad y^{kr} w = z_1^{\beta_1} \cdots z_t^{\beta_t} w.$$

Comparing it with (21) yields

$$(23) \quad y^{kr}w = x_1^{\alpha_1} \cdots x_s^{\alpha_s}w.$$

Applying (8) to (22) and (23) yields

$$(24) \quad y^{kr}z_1^{\beta_1} \cdots z_t^{\beta_t}w = z_1^{\beta_1} \cdots z_t^{\beta_t}w,$$

$$(25) \quad y^{kr}x_1^{\alpha_1} \cdots x_s^{\alpha_s}w = x_1^{\alpha_1} \cdots x_s^{\alpha_s}w.$$

Now, let $o(w)$ denote the number of variables occurring in w . If $s + o(w) \geq n - 1$, then (25) is a consequence of $Id_n(\mathbf{S})$, by Lemma 7.2(ii). Otherwise, (25) is among $Id_n(\mathbf{S})$. The same concerns (24).

From (24) and (25), by (8) (which is in $Id_n(\mathbf{S})$), we get (22) and (23), and these yield (21). Hence, (21) is a consequence of $Id_n(\mathbf{S})$, as required.

The proposition above improves Theorem 16 in [27], where the same was proved for $n = 2qk + 1$. Moreover, our bound is the best possible, as the following example shows.

7.5. Let \mathcal{V} be a theory of nil-semigroups defined by $x^k = 0$, and $\mathbf{F}_q = \mathbf{F}_q(\mathcal{V})$, the \mathcal{V} -free semigroup on q free generators. Then, $E(\mathbf{F}_q) \in \mathcal{N}(k, 0, 1)$, and no identity

$$x_1x_2 \cdots x_s = x_2 \cdots x_s$$

with $s < n = q(k - 1) + 2$ holds in \mathbf{F}_q . Indeed, suitably substituting powers of generators g_1, \dots, g_q for x_1, \dots, x_s would yield

$$g_1^k g_2^{k-1} \cdots g_q^{k-1} = g_2^{k-1} \cdots g_q^{k-1}$$

which is not true in \mathbf{F}_q (see Lemma 2.8). Similarly, one can prove that no nontrivial identity in less than n variables such that each variable occurs at most once on each side holds in \mathbf{F}_q . Using this, it is not difficult to show that (20), which holds in \mathbf{F}_q by Proposition 7.3, is not a consequence of $Id_{n-1}(\mathbf{F}_q)$.

7.6. Recall that a variety \mathcal{V} is said to be *finitely generated* if it is generated by a finite algebra (or equivalently, by a finite number of finite algebras). We prove the following.

7.7. **Theorem.** *Let \mathcal{V} be a variety of commutative semigroups with $m(\mathcal{V}) = m$ and $r(\mathcal{V}) = r$. Then, \mathcal{V} is finitely generated if and only if the identity (20) is in $E(\mathcal{V})$ for some $n \geq 2$.*

Proof. The “only if” part follows trivially from Proposition 7.3. For the “if” part, note that if $m > 0$, then, by Proposition 7.3, the \mathcal{V} -free semigroup $\mathbf{F}_n = \mathbf{F}_n(\mathcal{V})$ on n free generators generates \mathcal{V} . We show that the same is true for $m = 0$.

Since the identities in at most n variables holding in \mathbf{F}_n are precisely those belonging to $E(\mathcal{V})$, it is enough to show that every identity e holding in \mathbf{F}_n is a consequence of the set $Id_n(\mathbf{F}_n)$.

Suppose that e is given in the form of (21). If the number of variables $o(w)$ occurring in w is not less than $n - 1$, then the result is by Lemma 7.2(ii). Hence, assume that $o(w) = n - p$ with $p > 1$.

Let $v = x_1^{\alpha_1} \cdots x_{p-1}^{\alpha_{p-1}} x_p^{\alpha_p}$ be the word obtained from $x_1^{\alpha_1} \cdots x_s^{\alpha_s}$ by identification of variables with indices greater than $p - 1$, whenever $s > p$, and $v = x_1^{\alpha_1} \cdots x_s^{\alpha_s}$, otherwise. Analogously define the word u for $y_1^{\beta_1} \cdots y_t^{\beta_t}$.

Then, $vw = uw$ is a consequence of (21), and therefore is in $E(\mathbf{F}_n)$ (note, that $o(vw), o(uw) \leq n - 1$). Also, $x^\alpha w = y^\beta w$, obtained from (21) by identifications of all variables x_i and all variables y_i , respectively, is in $E(\mathbf{F}_n)$. Moreover, identifying separately x 's and y 's, we get that all $x_1^{\alpha_1} \cdots x_s^{\alpha_s} w = vw = x^\alpha w$, and $y_1^{\beta_1} \cdots y_t^{\beta_t} w = uw = y^\beta w$, are in $E(\mathbf{F}_n)$.

Now, by the assumption, the number of variables occurring in $x^\alpha w = y^\beta w$ does not exceed n . The same is true for $vw = x^\alpha w$. Further, by Lemma 7.2(ii) (or because it is trivial), $x_1^{\alpha_1} \cdots x_s^{\alpha_s} w = vw$ is a consequence of $Id_n(\mathbf{F}_n)$. Hence, $x_1^{\alpha_1} \cdots x_s^{\alpha_s} w = x^\alpha w$ is also a consequence $Id_n(\mathbf{F}_n)$. Similarly, $y_1^{\beta_1} \cdots y_t^{\beta_t} w = y^\beta w$ is a consequence of $Id_n(\mathbf{F}_n)$, and we conclude that so is (21).

The join of two finitely generated varieties is, of course, finitely generated, as well. From the theorem above, it follows that the intersection of two finitely generated varieties of commutative semigroups is also finitely generated. Hence, we have the following.

7.8. Corollary. *The set $\mathcal{L}(\text{Com}_F)$ of finitely generated varieties of commutative semigroups is a (not complete) sublattice of $\mathcal{L}(\text{Com})$.*

7.9. In [32] Sapir formulated ten basic questions concerning varieties, and considered them for varieties of semigroups. He demonstrated that, in general, varieties of semigroups have *none* of the nice properties suggested in his questions. To conclude our paper, we demonstrate that varieties of commutative semigroups enjoy *all* these properties. Below, we just state a suitable theorem and a corollary; for the very formulation of the questions the reader is referred to [32].

7.10. Recall that a variety is called *small* if its lattice of subvarieties is finite. If this variety is, in addition, finitely generated and finitely based, then it is called *Cross*.

Observe that for a sequence \mathbf{a} of the form $(1, 1, \dots, 1)$ there are only finitely many filters $J \supseteq [\mathbf{a}]$, and therefore, in view of Examples 2 and 5 in §6, a variety \mathcal{V} with $E(\mathcal{V}) = E(J, m, r, \pi)$ and $\mathbf{a} \in J$ is small. We show that the converse holds, as well.

Indeed, given small \mathcal{V} , consider the sequence of equational theories $E_n = E(\mathcal{V}) \vee \{x_1 \cdots x_n = x_1^{r+1} x_2 \cdots x_n\}^*$, $r = r(\mathcal{V})$. This sequence has to be finite; hence, $E_n = E_{n+1}$ for some $n \geq 1$. It follows that the identity $e : x_1 \cdots x_n = x_1^{r+1} x_2 \cdots x_n$ is a consequence of $x_1 \cdots x_{n+1} = x_1^{r+1} x_2 \cdots x_{n+1}$ and the identities in $E(\mathcal{V})$. Applying Lemma 5.2, we infer easily that the identity e is in $E(\mathcal{V})$, as required.

Note, that this argument holds for a variety with the ascending chain condition, as well.

It follows, in view of Theorem 7.7, that every small variety is finitely generated, and whence, in view of Proposition 1.2, Cross.

On the other hand, Higgins [14] proved that a variety $\mathcal{V} \in \mathcal{L}(\text{Com})$ is *saturated* in the sense of Howie and Isbell [16] (i.e., all epis are onto) if and only if a nontrivial identity of the form $x_1 \cdots x_n = (x_1 \cdots x_n)^k$ holds in \mathcal{V} .

Combining all these remarks we have the following.

7.11. Theorem. *For a variety \mathcal{V} of commutative semigroups the following conditions are equivalent.*

- (i) \mathcal{V} is Cross.
- (ii) \mathcal{V} is small.
- (iii) The ascending chain condition holds for the subvarieties of \mathcal{V} .
- (iv) \mathcal{V} is saturated.
- (v) A nontrivial identity of the form $x_1 \cdots x_n = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ holds in \mathcal{V} for some $n \geq 1$.

Following Sapir [32], call a sublattice of a lattice of varieties *stable* if it is closed under taking covers and subvarieties. A *just non-Cross* variety is a variety which is not Cross, but all of its proper subvarieties are Cross.

While there are infinitely many just non-Cross varieties of semigroups [32], the only known example of a just non-Cross variety of commutative semigroups is that of the variety $\mathcal{V}(x^2 = 0)$ defined by the law $x^2 = 0$ [35]. Using again arguments based on Lemma 5.2, we get easily the following.

7.12. Corollary. *The commutative semigroup Cross varieties form a stable sublattice of $\mathcal{L}(\text{Com})$; the variety $\mathcal{V}(x^2 = 0)$ is the only just non-Cross variety of commutative semigroups.*

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